# SINGLE MACHINE SCHEDULING WITH UNCERTAIN PROCESSING TIMES

BY

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# Dedication

*I dedicate this thesis to my beloved family for their support and encouragement.*

# ACKNOWLEDGMENTS


In the name of Allah, the Most Beneficent, the Most Merciful

All praise and glory goes to Almighty Allah (the Powerful and Exalted in Might) who gave me the courage and patience to carry out this work. Peace and blessings of Allah be upon his last prophet Muhammad (Peace and blessings be upon him) and all his companions (May Allah be pleased with them all).

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# THESIS ABSTRACT

**Name:** **Khaled Hashim Al-Shareef**

**Title:** **Single Machine Scheduling with Uncertain Processing Times**

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In this thesis a single machine scheduling problem is considered, where job processing times are uncertain with known upper and lower bounds of its actual value. The number of jobs is $n$ and they all have equal weights. All jobs are assumed to be ready at the beginning (time 0). The machine is continuously available over the planning horizon. All data other than processing times are deterministic and known in advance. The objective is to characterize the schedule that minimizes the completion time-related objective functions and the objective functions related to total earliness and tardiness.

Schedules that minimize the upper bound, the lower bound, and the range of the mean completion time are characterized, and they are identified under certain conditions. Similar results are derived when a third estimate of the processing time (most likelihood) is available.

In constrained scheduling, the due date is known. Schedules that minimize total earliness, total tardiness, total earliness and tardiness, and the ranges of total earliness and tardiness are characterized. These schedules are identified under some conditions. Similar results are derived when a third estimate of the processing time (most likelihood) is available. Then, a method to find the optimum due date is presented.

Complete enumeration is used to generate all possible schedules for a specific example by considering the lower and upper bounds to confirm the derived results.

**Keywords: single machine scheduling, uncertain processing times, common due date, mean completion time, and total earliness and tardiness.**

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# ملخص الرسالة

**الاسم:** **خالد هاشم الشريف**

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**تاريخ التخرج:** **يوليو ٢٠٠٨ م**


موضوع هذه الرسالة هو جدولة العمليات على جهاز واحد عندما تكون متطلباتها الزمنية معروفة مسبقا و كل ما هو معلوم عنها هو حدها الأعلى و حدها الأدنى. إلا أن عدد هذه العمليات معروف سلفا و هي جاهزة للتنفيذ منذ بداية الفترة الزمنية المحددة للجدولة مفترضين عدم حصول أعطال أو أعمال صيانة للجهاز خلال تلك الفترة الزمنية.

و يهدف هذا البحث إلى تحقيق مجموعة من الأهداف المتعلقة بأوقات إنهاء العمليات الصناعية و كذلك الأهداف المتعلقة بإنهاء العمليات في الموعد المحدد لها دون تأخير أو تبكير. و من ذلك إيجاد خصائص الجدوال التي تحقق أقل معدل لأوقات إنهاء العمليات بحدها الأدنى و حدها الأعلى و كذلك في المدى معرفا بالفرق بين الحدين الأعلى و الأدنى. كما تم تطوير نتائج مماثلة في حالة توفر معلومة إضافية عن الفترة الزمنية المتطلبة للعمليات الصناعية و هي الوقت الأكثر احتمالا.

أما في حالة الجدوال المقيدة بوقت انتهاء (تسليم) محدد سلفا فقد تم تحديد خصائصها التي تعطي أقل الأوقات المبكرة لكل العمليات و أقل الأوقات للعمليات المتأخرة و لمجموع كل الأوقات التي ليست في موعدها سواء تبكيرا أو تأخيرا و في بعض الحالات تم إيجاد طريقة لمعرفة الجداول كاملة. نتائج مشابهة تم الحصول عليها عند توفر معلومة إضافية عن الفترة الزمنية المتطلبة للعمليات و هي الزمن الأكثر احتمالا. بعد ذلك تم ايجاد طريقة لمعرفة الموعد المثالي للتسليم (الانتهاء) في حالة اتباع جدول زمني محدد.

في نهاية هذه الرسالة تم عمل برنامج حاسوبي لإيجاد و تقييم جميع الجدوال الممكنة و طبق على مثل رقمي محدد باستخدام الحدين الزمنيين الأدنى و الأعلى لتأكيد النتائج النظرية التي تم التوصل إليها في هذه الرسالة.

**الكلمات المهمة:** جدولة العمليات ' متطلبات زمنية غير محددة ' مواعيد التسليم الموحدة ' معدل أوقات انتهاء العمليات ' محصلة مقدار التبكير و التأخير.

درجة ماجستير في العلوم الهندسية

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يوليو ٢٠٠٨ م

# NOMENCLATURE

In developing models throughout this thesis, the following symbols are used:

| | | |
|---|---|---|
| $C_i$ | = | Completion time of job i |
| $P_i$ | = | Processing time of job i |
| $d$ | = | Common due date |
| $U_i$ | = | The upper bound of processing time for job i |
| $L_i$ | = | The lower bound of processing time for job i |
| $n$ | = | The number of jobs |
| $m$ | = | Most likelihood estimator |
| $E_i$ | = | The earliness of job i |
| $T_i$ | = | The tardiness of job i |

# CHAPTER 1

# Introduction

## 1.1 Background

Scheduling is defined in many ways, but the most famous definition is the allocation of resources to tasks over time with the goal of optimizing one or more given objectives [26]. Those resources include the use of equipment, the utilization of raw material or intermediates, machines in a workshop, runways at an airport, crews at a construction site, processing units at a computing office, or any similar uses. The tasks can be operations in a production process, take-offs and landings at an airport, stages in a construction project, execution of computer programs, and so on. However, the resources have certain capabilities, and the tasks have certain requirements, that must be satisfied for any proposed schedules to be feasible. The schedules are usually represented by using a Gantt chart that has two different orientations, machine-oriented and job-oriented [26]. A great variety of aspects need to be considered when developing scheduling models. The basis of scheduling is a proper description of the production process [6].

Any scheduling problem is described by the notation [26]:

$$\alpha / \beta / \gamma$$

$\alpha$ field describes the machine environment;

$\beta$ field includes any specific characteristics of the problem related to relations between jobs or the processing characteristics and constraints;

$\gamma$ field describes the objective to be minimized.

Many classifications of schedules can be found in the literature, and one of them is as follows: A sequence refers to the order in which jobs are to be processed on a given machine or a single machine. A schedule involves the sequence of jobs as well as timings. A scheduling policy describes an appropriate action for the states of the system. A non-delay schedule is a feasible schedule with no machine kept idle while an operation is waiting for processing. Non-preemptive non-delay schedules are non-delay schedules with no preemption. An active schedule is a feasible schedule that cannot be modified (by changing the order of operations) without delaying some operations. A semi-active schedule is a feasible schedule that cannot be modified (without changing the order of operations) without delaying some operations [26].

The single machine scheduling problem has been comprehensively studied for various performance measures. Most of these studies deal with the static scheduling where all jobs are available in advance. In addition, they consider either the deterministic case where job attributes (such as setup times, processing times, due dates) are known with certainty or the stochastic case where some of these attributes are assumed to be random variables [14].

## 1.2 Early-Tardy Problem

The single machine earliness–tardiness scheduling problem is well known in the literature, and it is of particular interest for many researchers because from an industrial point of view, commonly, only one machine can be the bottleneck in any process or a production line. Thus, good scheduling of jobs on that bottleneck machine will help the manufacturer in reducing the total earliness and tardiness of jobs along the process or production line. However, from an academic point of view, the single machine case is the simplest earliness–tardiness scheduling problem, and it is a special case for all other scheduling environments [10].

Nowadays the increasing competition and the challenging customer demands are pushing manufacturers to re-evaluate their manufacturing strategies. This is because they want to eliminate hidden production and inventory problems which cause high costs. Examples include set-up time, waiting time, delivery time, extra labor costs, rework, and order changes. Manufacturers should consider scheduling problems involving not only the tardiness penalty, but also the earliness penalty [9]. Many manufacturing systems, which adopt Just-in-Time (JIT) concepts, have created scheduling policies with the objective of producing each customer order at the due time, to prevent the direct and indirect costs incurred by completing jobs before or after their due dates [15]. Tardy jobs may incur tardiness costs, such as contractual penalties, depending on how late they are, while early jobs may possibly incur earliness costs, such as inventory holding costs, depending on how early they are. These earliness and tardiness costs are caused by the refusal of the products by customers, and by express deliveries or losses of sales regardless of how early or how late the jobs are [9]. However, in many real-world scheduling systems, both early and tardy jobs are penalized by unchanging penalties regardless of how early or tardy the jobs are. For example, in various industries, raw materials or parts are often needed to be available at specific times. Likewise, in air flight scheduling, tasks are to be performed on exact time points in order to ensure the success of a flight. Also, in pick-up and delivery systems, items must be picked up or delivered at certain times. Thus, when jobs are early or late, fixed penalties are incurred no matter how early or late the jobs are [14]. These costs lead to a problem whose objective is to minimize the sum of earliness and tardiness, known as the Earliness-Tardiness (E-T) problem. It is well known that this performance measure is non-regular [15]. Moreover, minimizing total earliness and tardiness can be crucial to the survival of several industries. For example, small and medium sized make-to-order apparel industries receive orders from their clients along with a specific due date for each order. These due dates are to be respected since they are dictated by shipping schedules and commitments of the clients [10].

## 1.3 Due Date Determination

Meeting a restrictive common due date is a general scheduling issue in practice. Such instances might occur if a customer orders different variations of a product, all of which must be delivered in the same truck or container ship for transportation cost saving, or if a firm has promised a weekly bulk delivery to the wholesaler. A survey of US manufacturing practitioners reveals that meeting a restrictive common due date is one of the most important scheduling criteria. Hence, in connection with JIT production and delivery, earliness and tardiness penalties are of continuing interest for scheduling researchers and practitioners. However, scheduling problems with a restrictive common due date where the objective is to minimize earliness–tardiness penalties is not a trivial issue. Even a relatively simple problem, such as the class of a single-machine, is non-deterministic polynomial-time hard (NP-hard) [21].

## 1.4 Uncertainty

Uncertainty in a real manufacturing situation is a complex phenomenon. Variability in processing speed has a different impact on the situation if the variation occurs early in the day or close to the end of a shift. Uncertainty that affects yield is more important after a few operations when value has been added and replacing the scrapped material in time to meet a due date is difficult, as opposed to yield variation in the very first operation. Operator performance may be more uncertain just before or after a long weekend than mid-week. Uncertainty experienced on the night shift may have more impact than the same uncertainty encountered during the day when additional support staff and management are available for problem-solving. There appear to be three key dimensions of uncertainty – cause, context, impact - that can help to categorize problem formulations and processes. For example, the cause may be tooling not available, the context is the bottleneck machine on Monday morning, and the impact is a delay in setup if the machine cannot start when expected. To explain the meaning of cause, context, and impact, the dimensions can be further analyzed. Cause can be viewed as object (e.g., material, process, resource, tooling, personnel) and state (e.g., ready, not ready, high quality, low quality, damaged, healthy). Context refers to the environmental situation at the time of the scheduled event (e.g., nothing special, resource just repaired or upgraded, when in week or day or shift (if it matters), experience or training of the crew). Essentially, is there anything about the context that would alter expectations for processing time, yield, or some other performance metric? The situation is either context-free or context-sensitive. A context-free situation would require no additional information or special decision making, whereas a context-sensitive formulation would have information about the context and associated implication. Impact refers to the result of the uncertainty. In general terms, the impact can be categorized as time, material, quality, independent or dependent, and context-free or context-sensitive [11].

Different sources of uncertainty can be recognized in production scheduling, from process uncertainties involving measurable data such as processing times or rates as well as discrete events such as equipment availability, to external uncertainties coming from environmental conditions, technology changes and market parameters. The main effects of these sources of uncertainty in terms of their impact on scheduled tasks are idle and waiting times. Waiting times can lead to unexpected delays, and may result in quality problems for sensitive or unstable materials. Moreover, idle times may result in equipment under-utilization and high inventory costs [17]. The prevalent approach to the treatment of these uncertainties is through the use of probabilistic models that describe the uncertain parameters in terms of probability distributions. However, the evaluation and optimization of these models is computationally expensive, either because of the large number of scenarios resulting from a discrete representation of the uncertainty, or the need to use complicated multiple integration techniques when the uncertainty is represented by continuous distributions. Furthermore, the use of probabilistic models is realistic only when these descriptions of the uncertain parameters are available, say from historic data. In such situations, we have to resort to an alternative treatment of uncertainty. For example, the modeler may be able to approximate the duration of the tasks and specify the longest and shortest durations or the interval in which the duration belongs at different levels of confidence [1].

Processing times are highly unpredictable in industries where custom-designed products are dominant. Inaccurate estimation of processing time exists in manufacturing and service industries, such as automobile repairing. The estimation error of processing time is considered as an important factor that affects the performance of scheduling policies. In the production literature, the estimation error of processing time has been addressed as processing time uncertainty. This uncertainty is different from the variation of actual processing time. The latter describes the situation that actual processing times vary from job to job. The processing time uncertainty (PTU) addresses the problems where processing times cannot be accurately estimated when jobs are scheduled [16].

Zukui and Ierapetritou (2007) summarized the ways of dealing with uncertainty and they described how it can be incorporated in the model. The summary of their work is as follows. Uncertainty in process operations can be generated from many aspects, such as demand or changes in product orders or order priority, batch or equipment failures, processing time variability, resource changes, recipe variations, etc… [6].

Based on the nature of the source of uncertainty in a process, a suitable classification was proposed by Pistikopoulos (1995) as follows:

(i) Model-inherent uncertainty, such as kinetic constants, physical properties, mass/heat transfer coefficients;

(ii) Process-inherent uncertainty, such as flow rate and temperature variations, stream quality fluctuations, processing time and equipment availability;

(iii) External uncertainty, including feed stream availability, product demands, prices and environmental conditions;

(iv) Discrete uncertainty, such as equipment availability and other random discrete events, operational personnel absence [2].

To include the description of uncertain parameters within the optimization model of the scheduling problem several methods have been developed:

(i) Bounded form:

In many cases, there is not enough information in order to develop an accurate description of the probability distribution that characterize the uncertain parameters, but only error bounds can be obtained. In the case of this thesis an interval can be used for uncertainty estimation. The interval represents the upper bound (most pessimistic estimate) and the lower bound (most optimistic estimate). In many cases, a third estimate is made, and that is the most likely estimate.

(ii) Probability description:

This is a common approach for the treatment of uncertainties when information about the behavior of uncertainty is available, since it requires the use of probabilistic models to describe the uncertain parameters.

(iii) Fuzzy description.

Fuzzy sets allow modeling of uncertainty in cases where historical (probabilistic) data are not readily available. The scheduling models based on fuzzy sets have the advantage that they do not require the use of complicated integration schemes needed for the continuous probabilistic models and they do not need a large number of scenarios as the discrete probabilistic uncertainty representations [1].

McCahon and Lee (1992) [8] were the first to illustrate the application of fuzzy set theory as a means of analyzing performance characteristics for a flow shop system. They modified the sequencing heuristic in Campbell et al. (1970) [13] for the case of fuzzy processing times. Most of the work in applying fuzzy set theory to scheduling optimization has primarily focused on using heuristic search techniques such as simulated annealing and genetic algorithms to obtain near-optimal solutions. Ozelkan & Duckstein (1999) [23] investigated the necessary conditions for optimality of fuzzy counterparts of classical (deterministic) optimal scheduling rules, and they emphasized the importance of using ranking functions that satisfy certain properties [1].

## 1.5 Motivation

The motivation for the present study came from our observations of the scheduling literature. For a single machine scheduling problem, the uncertainty consideration of processing times with the objectives of minimizing Total Completion Time (Mean Completion Time) and Earliness and Tardiness is overlooked. However, uncertainty is a very important concept in industry and service areas since not all parameters of the scheduling process can be determined exactly. Many parameters, such as raw material availability, prices, machine reliability, market requirements, and service time, vary with time, and they are subject to unexpected problems. In fact, the consideration of uncertainty in the scheduling problems has attracted more researchers in order to characterize the optimal solution. Moreover, bounded processing times are considered to be more realistic for many real-life problems and applications where the lower and upper bounds are the only available information about the problem and sometimes a third point representing the most likely estimate. The probabilistic approach assumes a certain distribution which is not necessarily representing the actual uncertainty.

## 1.6 Scope

The problem in this thesis is single machine scheduling under uncertainty in processing times, with respect to different performance measures related to job completion times and meeting due dates. These uncertainties in processing times are described by their upper and lower values. This problem is extended to the case where a third-point estimate of processing time is also known, and that is the most likelihood estimate.

## 1.7 Problem Definition

In JIT production systems, jobs are penalized for being early as well as tardy. As a result, early-tardy scheduling problems have increasingly attracted the attention of researchers. Raghavachari (1988) [28] and Baker and Scudder (1990) [27] provided reviews of the research done in this area, and they identified two types of early-tardy scheduling problems: constrained and unconstrained. In both problems, the final goal is to find a schedule that minimizes the sum of earliness and tardiness penalties. For problems with a common due date for all the jobs, the due date has a given fixed value in the constrained problem, whereas in the unconstrained problem the due date is a decision variable. The former problem is in general more difficult to solve. However, if the due date is un-restrictively larger than the desired optimum due date, then the two problems, constrained and unconstrained, are equivalent. The solution to the constrained problem is the schedule that solves the unconstrained problem with a delay in its starting time. In this latter problem, one typically determines an optimal due date value for a fixed schedule, and then finds a corresponding optimal schedule.

The problem that will be tackled in this thesis is the single machine problem considering uncertain processing times with different performance measures. Uncertainty of processing times is described by an upper and lower bound for each processing time.

These bounds are determined from data records of the same process or by experience if the records are not adequate.

This specific problem is not addressed in the literature. Most of the work done in this area considered uncertain (bounded) processing times in the flow shop and the job shop with various objectives. In fact, the single machine problem is considered as the basis for all scheduling problems, because it can be found in many industries and services, and it provides insight into the analysis of other more complicated scheduling problems.

### 1.7.1 Assumptions and Notations

These are the general assumptions and notations for this problem for all objectives:

- The environment of this problem is a single machine that is continuously available over the planning horizon.
- There are $n$ jobs, with uncertain processing times (for each processing time $P_i$ there is an upper bound $U_i$ and a lower bound $L_i$ known in advance).

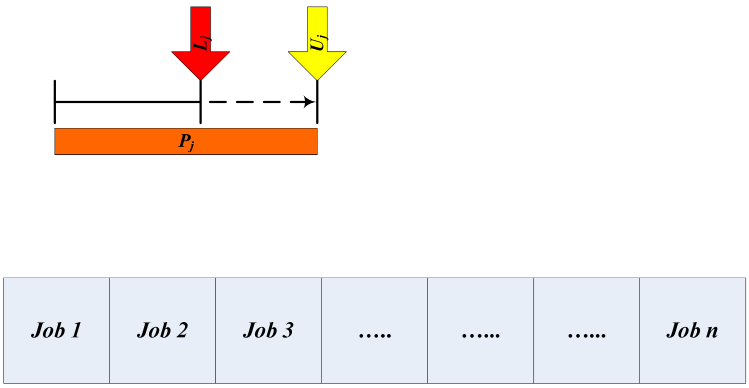



Figure 1-1: Bounded processing time.

- All jobs have the same priority, which means all weights are assumed to be equal.
- All jobs are assumed to be available at the beginning of the process *(at time 0)*.
- Once a job starts processing, it cannot be preempted.
- $P_{[j]}$: is the processing time of the job in the $j^{th}$ position.
- $C_{[j]}$: is the completion time of the job in the $j^{th}$ position.
- All data, other than processing times, are deterministic and known in advance.
- The notation $S$ denotes a complete sequence, whereas the notation $s$ denotes a subsequence.
- All jobs are independent from each other.

# CHAPTER 2

# Literature Review

The main reason behind the development of the scheduling theory is to solve problems like the ones occurring in production facilities. The basic scheduling problem can be described as finding, for each of the jobs, an execution interval on one of the machines that are able to execute it, such that all side-constraints are met. The resulting solution (schedule) is the best possible (optimal), that is, it minimizes the desired objective function. Since the first scheduling paper appeared in 1954, many variants and extensions of the basic scheduling problem have been formulated by differentiating between machine environments, side-constraints, and objective functions [5].

Arifusalam and Selim (1998) studied the single machine scheduling problem with interval processing times (given lower and upper limits) for minimizing several objectives. They tried to obtain the robust schedule for each objective they had. They defined the robust schedule as "the schedule which has the minimum worst case performance". Their first objective was to minimize the mean flow time. Their approach was by minimizing the lower and upper limits of mean flow time as separate objectives, and then minimizing the difference between the two previous limits. After that, they gave a method to obtain the robust schedule for mean flow time. The second objective was to minimize the maximum lateness (given that each job has a given due date). The third objective was to minimize the number of tardy jobs, and they used two approaches to achieve this objective. Then, the method of finding the robust schedule for this objective was given. The last objective was to minimize the sum of earliness and tardiness (with the assumption of having a common due date and equal job weights). They gave a method to find the common due date. After that, they generalized their work to include different job weights [72].

Hoogeveen (2005) showed that, until the late 1980s, it was common that only one performance criterion was taken into account in the objective function. In practice, however, quality is a multidimensional notion. In order to reach an acceptable compromise between these dimensions, one has to measure the quality of a solution on all important criteria. This observation led to the development of the area of multi-criteria scheduling. Probably the most appealing bi-criteria scheduling problem is the earliness–tardiness problem. The justification comes from the just-in-time philosophy in management theory: "the best time for getting an item delivered is the time at which it is needed, which time is given by the due date". As a result, when scheduling the production process, the manufacturer will be unhappy with both an early completion that results in the need to store the product until it can be shipped, and a late completion of a product that will decrease customers' satisfaction [5].

Suppose that two performance criteria have been selected, say $f$ and $g$. Without loss of generality, we assume that these criteria are to be minimized. Unless we are extremely lucky, there will be no schedule that achieves the minimum value for both performance criteria simultaneously, which implies that we have to accept a reduction in the quality of at least one of the two criteria. If one performance criterion, say *f*, is far more important than the other one, then the best approach is to find the optimum value with respect to criterion *f*, say $f^*$, and choose from among the set of optimum schedules for *f* the one that performs best on *g*. This approach is called hierarchical optimization or lexicographical optimization [5].

Much more work has been done on the problem of minimizing the function of total earliness and total tardiness. Du and Leung (1990) [42] showed that $1||\sum_j T_j$ is NP-hard. All the earliness–tardiness problems without simplifying assumptions are NP-hard, too [5].

Kanet (1981) [43] was the first researcher who introduced the area of common due date early-tardy scheduling. This has subsequently received a lot of attention, mainly due to the rich structure possessed by optimal schedules. Kanet looked at the problem $1|\ d_j = d\ |\ \sum_j (E_j + T_j)$. This objective function can be rewritten as $\sum_j | C_j - d_j |$ to represent the total deviation from the common due date d [5].

## 2.1 Early-Tardy problem

Scheduling is one of the research topics that scholars and practitioners have studied widely because the methodology for finding an optimal or good schedule can be applied to real situations [19]. For many years, scheduling research has been mainly focused on a regular measure which increases with the increase in job completion time. As the JIT production system prevails at many companies in the world, the earliness/tardiness (E-T) model has received much attention because the notion of just-in-time (JIT) is that the exact quality and quantity of products are produced in just the right place at just the right time. In the E-T model, jobs whose completion times do not meet the due date are penalized. The importance of the E-T measure, a non-regular measure, is mentioned in many papers such as Kanet (1981) [43]. Various E-T models are reorganized in Baker and Scudder (1989) [44] according to their objective functions (MAD: mean absolute deviation or MSD: mean squared deviation), their due dates (a common due date or a distinct due date), the number of machines (single or multi-machine), the weights on earliness and tardiness, etc. The first research to consider earliness and tardiness concurrently was by Merten and Muller (1972) [45]. Kanet (1981) [43] proposed a simple procedure that could solve the MAD problem in which weights on earliness and tardiness were equal when the given due date was greater than, or equal to, the makespan. Bagchi et al. (1986) [46] introduced the unconstrained problem and the constrained problem according to the tightness of the due date, i.e., whether or not the due date constrains the minimization of MAD value. They enlarged the due date bound for the unconstrained MAD problem. Hall et al. (1991) [47] proposed optimal conditions for the MAD problem when weights on earliness and tardiness are the same. They proved that the constrained E-T problem is NP-complete in the ordinary sense, and they described a case which can be solved in polynomial time [19].

Earliness and tardiness are measured in different ways, such as squared deviation and absolute deviation. Jobs may have different weights or equal weights. Earliness and tardiness may have equal or different weights. Furthermore, jobs can have different due dates or a common due date. Some researchers combine other criteria with the early-tardy criteria.

Another major classification of the early-tardy scheduling problem is whether the problem is deterministic or stochastic. The latter case involves an element of randomness, such as stochastic processing times or unreliable machines. Both deterministic and stochastic classes are discussed below in detail.

Most of the research in the early-tardy area assumes an ideal environment where processing times are known and machines are continuously available. Raghavachari (1988) [28] and Baker and Scudder (1990) [27] reviewed the research done in this area for both quadratic and absolute deviation measures.

Sidney (1977) [48] was probably one of the first researchers who studied the single machine scheduling problem with earliness and tardiness penalties. He considered the problem of minimizing the maximum earliness or tardiness penalty. Seidmann et al. (1981) [49] considered the problem of assigning individual job due dates and identifying a sequence so as to minimize weighted earliness and tardiness and lead time costs. They showed analytically that the optimal due date assignment is related to the completion time of the respective jobs or to the value of the specified lead time that customers consider to be reasonable. They also proved that the optimal solution is to schedule jobs in non-decreasing order of processing times with the first job starting at time zero. Garey et al. (1988) [50] showed that the single machine scheduling problem with distinct due dates, where the objective is to minimize the sum of the absolute deviation of job completion times from their corresponding due dates, is NP-complete. Several researchers developed procedures to solve earliness–tardiness scheduling problems assuming no inserted idle time. With regard to this problem, Abdul-Razaq and Potts (1988) [51] used a dynamic programming state space relaxation method by mapping the original state space onto a smaller state space, and they derived a procedure to obtain lower bounds by applying a recursive scheme on the smaller state space. Szwarc (1993) [52] studied this problem, and he provided criteria for when the problem is decomposable into separate smaller sub-problems. He presented a simple branching technique that generates schedules that cannot be improved by adjacent job interchanges. Yano and Kim (1991) [53] developed a dynamic programming procedure to find the optimal timing of jobs for a given sequence. They also developed optimal and heuristic procedures for a subset of problems where the weights are proportional to the job processing times [20].

There are two main approaches to address the common due date. In the unrestricted case, the common due date is a decision variable or, if its value is known, it has no influence on the optimal sequence. This happens when the due date is greater than or equal to the sum of all processing times. However, if the due date is known and it affects the optimal sequence of jobs, then it is considered restrictive. Kanet (1981) [43] was one of the pioneers for the unrestrictive case, while Cheng and Gupta (1989) [29] made a survey about models in which due dates are decision variables. For the restrictive common due date case with general penalties, there is an optimal solution with the following properties:

(1) No idle times are inserted between consecutive jobs (Cheng and Kahlbacher, 1991) [54].

(2) The schedule is V-Shaped, meaning that the jobs which are completed at or before the due date are sequenced in non-increasing order. Jobs whose processing starts at or after the due date are sequenced in non-decreasing order. Note that a straddling job can exist, i.e. a job whose processing starts before and finished after the due date (Biskup and Feldmann, 2001) [55].

(3) There is an optimal schedule in which either the processing time of the first job starts at time zero or one job is completed at the due date (Biskup and Feldmann, 2001) [55], [24].

The characterization of the optimal schedule as V-shaped reduces the set of possible schedules from the (n!) permutations to $2^{n-1}$ V-shaped schedules. If the optimal solution is a V-shaped schedule, then a simple dynamic programming algorithm in Kahlbacher (1992) can be used to construct a schedule which is optimal for a specified starting time [30].

Many authors studied the earliness and tardiness (E-T) scheduling problem. A comprehensive survey on the common due date assignment and scheduling problems can be found in Gordon et al. (2002) [24].

Feldmann and Biskup (2003) [56] studied the restricted E-T problem postponing the schedule by applying different metaheuristics: evolutionary search (ES), simulated annealing (SA) and threshold accepting (TA). Through a comparative analysis, the TA metaheuristic achieved better results in most of the tested problems in terms of computational processing time and quality of results [24].

Different methods are used to solve early-tardy problems. Since the problem, in most cases, is NP-complete, the focus of research is to identify the characteristics of an optimal schedule in order to reduce the size of the set of schedules that may be optimal. The most common characterization used in the early-tardy literature is the nonexistence of idle times between jobs. This characterization implies that a sequence of jobs and a start time for that sequence are sufficient to specify a schedule. Hence, it is sufficient to search for an optimal schedule in the set of permutation sequences. Identifying the starting time of an optimal schedule is another criterion that is commonly used.

The characteristics and dominance conditions obtained are embedded in some search techniques to solve different problems. Fry et al. (1987) used Branch and Bound methods [31], while Cheng (1990) used Dynamic Programming to search for an optimal schedule [32]. Mittenthal et al. (1993) used simulated annealing to search for a near optimal solution among V-shaped schedules [33].

### 2.1.1 Stochastic Early-Tardy Problem

The above research assumed an ideal manufacturing environment where processing times are known and machines are continuously available. These assumptions are not valid for most realistic processing environments. Typically machines are subject to breakdowns and job processing times are not always deterministic. Cheng (1984) studied the problem of squared deviation with random processing times with known means and the same coefficient of variation. He gave some properties of the optimal solution [34].

Recently, efforts have been made to consider stochastic scheduling problems. However, most of these efforts are devoted to studying regular performance measures. Birge et al. (1990) considered minimizing the expected total cost on a single machine, where cost is a non-decreasing function of completion times and the machine is subject to random breakdowns [35].

Stochastic problems with non-regular performance measures are rarely investigated. Vani and Raghavachari (1987) considered the situation where jobs have random processing times with the objective of minimizing the expected variance of completion times [57].

Krieger and Raghavachari (1988) considered minimizing the expected sum of absolute deviation from a common due date where jobs have stochastic processing times [58].

Cheng (1991) considered the problem of minimizing the expected squared deviation from a common due date where jobs have random processing times. Optimal due dates were analytically evaluated, and under certain conditions a polynomial algorithm was developed for the sequencing problem [36].

Mittenthal and Raghavachari (1993) studied the case where the single machine is subject to random breakdowns with the objective of minimizing the expected squared deviation from a common due date [37].

Federgruen and Mosheiov (1994) proved that the V-shape property holds even for the non-homogeneous Poisson process form of machine breakdowns. They have also derived general conditions for the early-tardy cost structure under which the V-shape property holds [38].

Soroush and Fredendall (1994) considered the cases where processing times are random and due dates are distinct for minimizing the total expected earliness and tardiness. They studied the case where processing times are normally distributed, and they proposed different heuristics for finding near-optimal solutions [39].

Soroush (1999) studied the problem of simultaneous due-date determination and sequencing of a set of n jobs on a single machine where processing times were random variables and job earliness and tardiness costs were distinct. He presented an analytical approach to determine optimal due-dates, and he proposed two efficient heuristics of order $O(n \log n)$ to find candidates for the optimal sequence. He demonstrated that variations in processing times increase the cost and the affect sequencing and due-date determination decisions [40].

Portougal et al. (2006) studied the statistically independent random processing times for a single machine without idling between jobs and without preemption. Their problem was to set due dates and promise them to customers during the production stage with the goal of minimizing the total expected penalties. They considered two due-date setting procedures with optimum customer service level, and an O (n log n) time complexity. They showed that one is asymptotically optimal but the other is not. Both heuristics included safety time, and the sequence remained the same regardless of disruptions, so that the result was robust. For the normal distribution, they provided sufficient optimality conditions, precedence relationships that the optimal sequence must obey, and tight bounds [41].

## 2.2 Due date Determination

In this part, a unified framework of the common due-date assignment problems in the deterministic case is presented by surveying the literature concerning the models involving a single machine. In early papers, the computer simulation techniques were applied to determine the better due dates (Conway, 1965; Eilon and Chowdhury, 1976; Weeks and Fryer, 1979) [22]. Seidmann et al. (1981) and Panwalkar et al. (1982) were the first who considered the optimal due-date assignment problems together with scheduling decisions and tackled the problems analytically [22].

The problems with due-date determination have received considerable attention in the last 15 years because of the introduction of new methods of inventory management such as just-in-time (JIT) concepts. In JIT systems, jobs are to be completed neither too early nor too late, which leads to the scheduling problems with both earliness and tardiness costs and assigning due dates. Cheng contributed to the due-date assignment and the related scheduling approaches, remarking that ‘‘completing a job early means to bear the costs of holding unnecessary inventories, while finishing a job late results in contractual penalty and loss of customer goodwill’’ [22].

Cheng (1986) initiated the studies with stochastic job processing times. Cheng (1991) considered a distinct due-date assignment and job sequencing problem where the processing time of each job is a random variable with known mean and variance but no knowledge of the specific distribution. The objective was to minimize the penalty associated with the deviation of the completion time of each job from its due date and the penalty of assigning late due dates [36]. However, the model implicitly assumed an equal penalty for earliness and tardiness. Some recent researches, such as Al-Turki et al. (1996) [59] took into consideration the possibility of machine breakdowns. For the common due date assignment problem, they either proved that the V-shaped sequence is optimal or they provided conditions for the conclusion to be true for several objective functions [25]. In the due-date determination problem, for a given schedule, the requirement is to assign a due date for each job that minimizes the cost function.  This problem is usually studied as a first stage in the unconstrained studies. However, some research was conducted purely on the due date determination problem, such as Seidmann et al. (1981).

Cheng and Gupta (1989) [29] reviewed the research in the due-date determination problem. They made a distinction between two types of due-date assignment procedure; exogenous and endogenous. These two types are used mainly for dynamic scheduling problems.

In the exogenous procedures, due dates are fixed and given attributes of the job. Two exogenous procedures are identified: Constant (CON), where jobs are given exactly the same allowance, and Random (RAN), where the flow allowance is randomly assigned. Among the researchers using CON in the early-tardy problem are Cheng (1988, 1991), and Bector et al. (1988). Seidmann et al. (1981) presented an analytical approach for a single machine scheduling problem with CON due-date assignment.

In the endogenous procedure, due dates are set by the scheduler. Examples of endogenous procedures are: TWK, where due dates are based on total work content; SLK, where jobs are given flow allowances that reflect equal waiting times or equal slacks; and NOP, where due dates are based on the number of operations to be performed on the jobs. In most unconstrained early-tardy scheduling problems, SLK or TWK procedures are used [29].

## 2.3 Scheduling under Uncertainty

Uncertainty and the disruptions associated with the resulting perturbations have been topics of discussion since the early 1900s. For example, Gantt (1919) is known for what is called the Gantt Chart, but he developed several different charts and the one that he considered the most useful was not the planning chart, but the chart prepared by the floor workers (operators or supervisors) providing feedback to the planners and schedulers. An early description of the scheduling task explicitly noted that the planners had to anticipate future difficulties and discount them (Coburn, 1918). Disruptions and uncertainty have been a problem since the beginning of systemized manufacturing, and they remain so today [11].

There has been an extensive body of research on production scheduling problems since their original mathematical formulation in the late 1950s. This literature can be broadly classified into two main areas: deterministic scheduling, where all problem parameters are assumed to be known with certainty; and stochastic scheduling, where at least some parameters are random variables. Much of the stochastic scheduling work has assumed that all parameters are random variables, and it has focused on local control policies such as dispatching rules aimed at minimizing some measure of performance in the expectation. Most of these methods use information about the global state of the shop, or they try to create a schedule for the entire shop prior to its execution. In deterministic scheduling research, a larger view is taken and multiple machines are often modeled. The deterministic approach is to plan the work through the machines over a period of time in the best way possible, given a specific objective to optimize. In recent years, many authors recognized that executing schedules directly as developed is an unlikely scenario in many manufacturing applications, and they have made efforts to extend the deterministic approaches to situations with some form of uncertainty. The basic assumption in much of this research is that a system that works in a deterministic environment can be engineered to work under at least certain stochastic conditions. A pervasive assumption in the deterministic scheduling field has been that the schedule, once released to the production floor, can be executed as planned. However, many production systems are subject to executional uncertainties that prevent the execution of a production schedule exactly as it is developed. Examples of such disruptions include machine failures, quality problems, arrival of urgent jobs, and a myriad of other possibilities. The inability of much scheduling research to address the general issue of uncertainty is often cited as a major reason for the lack of influence of scheduling research on industrial practice [11].

The issue of robustness in scheduling under uncertainty has received relatively little attention, in spite of its importance and the fact that a substantial amount of research addresses the problem of design and operation of batch plants under uncertainty. Most of the existing work has followed the scenario-based framework, in which the uncertainty is modeled through the use of a number of scenarios, using either discrete probability distributions or the discretization of continuous probability distribution functions, and the expectation of a certain performance criterion, such as the expected profit which is optimized with respect to the scheduling decision variables. Bassett et al. (1997) [60] considered process uncertainties in processing time fluctuations, equipment reliability/availability, process yields, demands, and manpower changes. They used Monte Carlo sampling to generate random instances, they determined a schedule for each instance, and they generated the distribution of aggregated properties to infer operating policies. Sanmarti et al. (1997) [61] presented a different approach for the scheduling of production and maintenance tasks in multipurpose batch plants in the face of equipment failure uncertainty. Because of the significant difficulty in the rigorous solution of the resulting problem, a heuristic method was developed to find solutions that improve the robustness of an existing schedule. There have also been attempts to transform a stochastic model to direct deterministic equivalent representation. Orçun et al. (1996) [62] considered uncertain processing times in batch processes, and they employed chance constraints to account for the risk of violation of timing constraints under certain conditions, such as uniform distribution functions [18].

Although a substantial amount of research addresses the problem of designing and planning under uncertainty, the issue of uncertainty in scheduling problems has received relatively little attention. Ierapetritou and Pistikopoulos (1996) [63] addressed the scheduling of single-stage and multistage multiproduct continuous plants with a single production line at each stage when uncertainty in product demands is involved. They used Gaussian quadrate integration to evaluate the expected profit, and they formulated the problem as a MILP model.

Lin et al. (2004) [18] proposed a robust optimization method to address the problem of scheduling with uncertain processing times, market demands, or prices. The robust optimization model was derived from its deterministic model by considering the worst-case values of the uncertain parameters, and a certain infeasibility tolerance was introduced to allow constraint violations.

Vin and Ierapetritou (2001) [64] addressed the problem of multiproduct batch plant scheduling under demand uncertainty. They introduced a robustness metric based on deviations from the expected performance including the infeasible scenarios. Robust schedules are generated on the basis of a multi-period approach.

Balasubramanian and Grossmann (2002) [65] considered uncertain processing times in the scheduling of multistage flowshop plants. They also proposed a multi-period MILP model, and they proposed a special branch-and-bound (B&B) algorithm with an aggregated probability model to select the sequence of jobs with minimum expected makespan.

Bonfill et al. (2005) [66] used a two-stage stochastic approach to address the robustness in scheduling batch processes with uncertain operation times. The objective was to minimize a weighted combination of the expected makespan and wait times.

Jia and Ierapetritou (2004) [67] developed a B&B solution framework to determine a set of alternative schedules for a given range of uncertain parameters. They employed the idea of inference-based sensitivity analysis for MILP problems, which has the advantage of not substantially increasing the complexity compared with the deterministic formulation.

Cheng et al. (2003) [68] presented an interesting comparison of optimal control and stochastic programming from a formulation and computation perspective. They showed that, although approximate dynamic programming leads to suboptimal solutions, it is much more computationally efficient, and thus can be used to address issues of nonlinearities and infeasibilities associated with mathematical programming [12].

Most of the work done with the uncertainty assumption considers the flow shop environment. Allahverdi (2006) mentioned that most research on flow shop scheduling problems assumes that job processing times have known and fixed values. There are many real-life problems in which this assumption is true, such as classes' times in the university [7]. However, this cannot be assumed in other scheduling problems. For these, the job processing times are unknown variables, and the only information that can be obtained is about lower and upper bounds for each job, namely bounded processing times.

Allahverdi and Sotskov (2003) addressed the two-machine flow shop scheduling problem to minimize makespan when job processing times are bounded [3]. Sotskov et al. (2004) addressed the same problem but with total completion time criterion [4]. Both of these studies assumed that setup times are included in processing times.

Most scheduling formulations belong to the class of NP-complete problems, even when simplifications in comparison to practical problems are introduced. The inclusion of uncertainty in process scheduling problems transforms the original deterministic model to stochastic or parametric formulations, which make the problem more complicated. In the direction of stochastic scheduling, reducing the computational cost is still a major issue.

The use of problem-specific heuristics can lead to efficient solution procedures. Such heuristic algorithms incorporate specific knowledge which often leads to good solutions in an acceptable amount of time. However, the use of heuristics has the disadvantage that they cannot usually guarantee convergence and good quality solutions [6].

The processing time uncertainty (PTU) was classified in the literature as the stochastic scheduling models that treat job attributes as independent random variables with given distributions, whose actual values are realized only after a scheduling decision has been made (Wein and Ou, 1991; Daniels and Kouvelis, 1995). Many researchers noticed the concept of PTU. Some researchers (Wein and Chevalier, 1992; Melnyk et al., 1989; Ishii and Muraki, 1996) formulated PTU in their simulation models, though they did not use it as an experimental factor for investigation of its effect on the performance of scheduling policies. More researchers used it as an experimental factor, and they found a significant effect of PTU (Baker and Dzielinski, 1960; Wein and Ou, 1991; He et al., 1994; Daniels and Kouvelis, 1995; Lawrence and Sewell, 1997; Sabuncuoglu and Karabuk, 1999; Cao et al., 2001). However, other researchers found the effect was trivial [16].

Cao et al. (2001) [69] noticed and mathematically proved that in some previous studies the operational definitions of PTU confounded the PTU with the variation of actual processing time as one factor. That is, the variation of actual processing time increased along with the levels of estimation error. Although statistically significant and/or practically substantial effects were found, the internal validity was too weak to attribute the poor performance to PTU [16].

# CHAPTER 3

# Mean Completion Time-related Performance Measures

In this chapter, the objective is to minimize the total completion time or equivalently mean completion time $\overline{C}$. This objective is related to the waiting time of the jobs and to reducing in-process inventory. Minimizing $\overline{C}$ for a single machine problem, when processing times are known in advance, can be achieved by the SPT rule in which all jobs are ordered in the increasing order of their processing times. However, when the processing times are uncertain, the problem becomes more complicated, since the SPT order is not known in advance. If we assume that jobs follow a certain distribution with a known expected value, then we can minimize the expected mean completion time by ordering jobs in the SPT order with respect to their expected processing times. In our problem, the processing times are unknown, and their expected values and distribution functions are also unknown. This makes the problem even more difficult. Another difficulty is that the completion time of any job cannot be determined until the job is actually processed, and if we want to describe them we must use their expected value or its minimum or maximum values. Thus, if we want to describe them by a certain criterion, we may in that case use the term “with probability 1”. In this chapter, we will derive some of the characteristics of the optimal solution, and we will construct schedules for objective functions related to it.

## 3.1 Formulation

The objective function for any given schedule can be written as follows:

$$\sum_{i=1}^{n} C_i = C_1 + C_2 + C_3 + \ldots.. + C_n = C_{[1]} + C_{[2]} + \ldots. + C_{[n]}$$

*where*

$$C_{[1]} = P_{[1]}$$
$$C_{[2]} = P_{[1]} + P_{[2]}$$
$$C_{[3]} = P_{[1]} + P_{[2]} + P_{[3]}$$
$$.$$
$$.$$
$$C_{[n]} = P_{[1]} + P_{[2]} + P_{[3]} + \ldots\ldots + P_{[n]}$$

$$\sum_{i=1}^{n} C_i = P_{[1]} + \left(P_{[1]} + P_{[2]}\right) + \left(P_{[1]} + P_{[2]} + P_{[3]}\right) + \ldots. + \left(P_{[1]} + P_{[2]} + P_{[3]} + \ldots + P_{[n]}\right)$$
$$= nP_{[1]} + (n-1)P_{[2]} + (n-2)P_{[3]} + \ldots + (n-k+1)P_{[k]} + \ldots + P_{[n]}$$

Therefore, the total completion time can be written as

$$\sum_{i=1}^{n} C_i = \sum_{i=1}^{n} (n-i+1)P_{[i]} \qquad (1)$$

Here, $P_{[i]}$ have uncertain values within a range $[L_{[i]}, U_{[i]}]$ and similarly for $C_{[i]}$'s

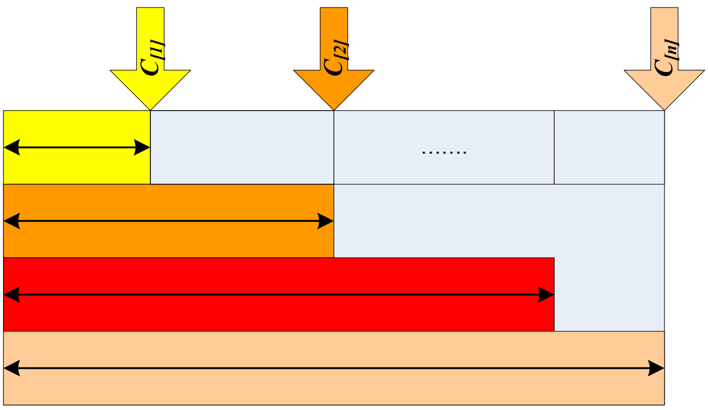



Figure 3-1: The range of completion times.

The objective function here is expressed in terms of the ordered processing times, and it will be used to characterize the optimal solution. The mean completion time is a regular performance measure, and it is expected that the optimal schedule maintains the usual characteristics associated with such measures.

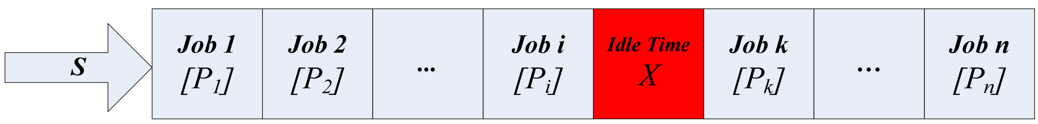
The following theorem proves that in this problem there is no idle time between jobs, and this can be explained easily from the environment of the problem, since we are working with one machine problem.

## Theorem (3.1)

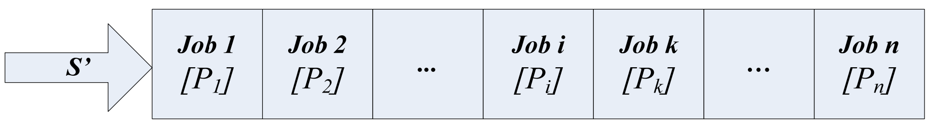
In this problem, $1/(\text{uncertain processing times})\ L_i \le P_i \le U_i / \sum_{i=1}^{n} C_i$, there is an optimal solution in which no idle time is inserted between jobs.

## Proof

Let us assume that $S$ is an optimal schedule with an idle time of length $X$ inserted between two jobs as follows:




And let $S'$ be the resulting schedule after removing the idle time from $S$ as follows:




Clearly $C_i' \le C_i$ for all $i = 1, 2, ..., n$, where $C_i$ and $C_i'$ are completion times of jobs in $S$ and $S'$ respectively.

$$C_j^{'} = C_j - X \qquad \text{where } X \text{ is constant}$$
$$= \sum_{i=1}^{j} P_i - X \qquad \text{where } \sum_{i=1}^{j} P_i \text{ is uncertain}$$

$$C_j = C_j^{'} \qquad \text{for all } j = 1,2,...,i$$
$$C_j \geq C_j^{'} \qquad \text{for all } j = k, k+1,...,n$$

The completion times for all jobs preceding job *k* remain unchanged, $C_j = C_j^{'}$ while the completion times for all jobs after job *k* (including job *k*) are smaller by *X*, $C_j \geq C_j^{'}$. This is true regardless of the fact that $P_i$s are uncertain, since actual processing times are independent of the actual time of processing.

Characterizing the optimal solution by having no inserted items (idle times) reduces the solution space from an infinite space to a finite space. In this case (single machine), there are (*n!*) possible sequences with a unique schedule corresponding to each sequence.

In the next theorem, one of the dominance relations is proved. In this relation, the job with certain smaller processing time precedes the job with certain larger processing time. Since the processing times are uncertain, the upper and lower bounds are used for making the comparison. This theorem proves that, if we have two jobs, and if one of them has a lower bound that is greater than or equal to the upper bound of the other job, then the job with smaller upper bound should be processed before the job with the larger lower bound in any optimal schedule. Under this condition, the relationship between the two processing times is known with certainty. Such pairwise relationships between jobs in the optimal schedule further reduce the solution space from (*n!*) to a smaller size that may be easier to search and embedded in different solution methods.

## **Theorem (3.2)**

For any two jobs *i* and *k* with $U_k \leq L_i$, there is an optimal schedule in which job *k* precedes job *i*.

### **Proof**

Let us assume that $S = (s_1, i, s_2, k, s_3)$ is an optimal schedule in which $U_k \leq L_i$ and *job i* precedes *job k*, in position *x* and *job k* in position *y*, where $x < y$ and $S' = (s_1, k, s_2, i, s_3)$ is another schedule that results from interchanging jobs *i* and *k* in *S*.

The objective function for the sequence *S* is as given in (1):

$$\begin{aligned} f(S) &= \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} (n-i+1)P_{[i]} \\ &= nP_{[1]} + (n-1)P_{[2]} + (n-2)P_{[3]} + \ldots. + (n-x+1)P_i + \ldots. + (n-y+1)P_k + \ldots. + P_{[n]} \end{aligned}$$

The objective function for the sequence *S'* is:

$$\begin{aligned} f(S') &= \sum_{i=1}^{n} C_i = \sum_{i=1}^{n} (n-i+1)P_{[i]} \\ &= nP_{[1]} + (n-1)P_{[2]} + (n-2)P_{[3]} + \ldots. + (n-x+1)P_k + \ldots. + (n-y+1)P_i + \ldots. + P_{[n]} \end{aligned}$$

The difference between the objective functions of the two schedules is derived as follows:

$$f(S) - f(S') = [(n-x+1)P_i + (n-y+1)P_k] - [(n-x+1)P_k + (n-y+1)P_i]$$

$=(x-y)P_k + (y-x)P_i$

$=(y-x)(P_i - P_k) \geq 0$ ( positive with probability 1)

Since $y > x$ and $P_i \geq P_k$ with probability 1 (since $U_k \leq L_i$).

Since $S$ is optimal, then $S'$ is also optimal.

Assuming uncertain processing times made it difficult to minimize $\overline{C}$ directly. This is because we do not have any information about these processing times, except the upper and lower bounds. Therefore, we tried to find another way to minimize $\overline{C}$ indirectly. One of these ways is to minimize the maximum $\overline{C}$ and minimum $\overline{C}$. Finding maximum $\overline{C}$ is not difficult since we are dealing with bounds and so, if all processing times took the upper bound values, then we will have the maximum $\overline{C}$ and the following theorem proofs how to minimize this value.

In the following theorem, we will characterize the optimum schedule that minimizes the upper bound of the total completion time $UB\ (\sum C_i)$ or equivalently the upper bound of the mean completion time $\overline{C}_u$. The resulting schedule will reduce the risk of having a high total completion time.

## Theorem (3.3)

To minimize $\overline{C}_u$, jobs should be ordered in increasing order of their upper bounds ($U_i$).

**Proof**

For any given sequence, the total completion time is defined, from(1), as :

$$\sum_{i=1}^{n} C_i = \sum_{i=1}^{n} (n-i+1) P_{[i]}$$

For any given sequence, the highest possible value that $\sum C_i$ can take is when the processing times of all jobs have their highest possible values (upper bounds).

Therefore

$$UB\left(\sum_{i=1}^{n} C_i\right) = \sum_{i=1}^{n} (n-i+1) U_{[i]}$$

Thus, minimizing $\overline{C}_u$ is equivalent to minimizing $\sum_{i=1}^{n} (n-i+1) U_{[i]}$ which can be achieved by ordering jobs in non-decreasing order of the $U_i$s (matching the highest coefficient with the smallest U).

Minimizing the maximum mean completion time deals with the schedule with the minimum worst case performance in terms of mean completion time. This measure is quite useful when the cost of job delay is much higher than the benefit gained by completing jobs early in the schedule. The decision maker in this case would prefer a schedule that reduces the risk of delays in job completion times.

Another performance measure related to $\overline{C}$ is its minimum ($\overline{C}_l$). Minimizing $\overline{C}_l$ is needed when the decision maker is willing to take the risk of possible delay for the possibility of completing jobs in the earliest possible time. This is because customer satisfaction has the priority over the in-process inventory. A schedule that minimizes the lower bound of the total completion time increases the chance of obtaining low total completion time *LB* ( $\sum C_i$ ).

## Theorem (3.4)

To minimize $\overline{C}_l$, jobs should be ordered in the increasing order of their lower bounds ($L_i$).

### Proof

For any given sequence, the total completion time is defined, from(1), as :

$$\sum_{i=1}^{n} C_i = \sum_{i=1}^{n} (n-i+1) P_{[i]}$$

The lowest possible value that $\sum C_i$ can take is when the processing times of all jobs have their lowest possible values (lower bounds).

Therefore

$$LB\left(\sum_{i=1}^{n} C_i\right) = \sum_{i=1}^{n} (n-i+1) L_{[i]}$$

Thus, minimizing $\overline{C}_l$ is equivalent to minimizing $\sum_{i=1}^{n} (n-i+1) L_{[i]}$ which can be achieved by ordering jobs in non-decreasing order of the $L_i$s (matching the highest coefficient with the smallest L).

Another measure related to $\overline{C}$ is minimizing its range, and this theorem shows how to minimize the range of $\overline{C}$ by arranging jobs with their ranges in increasing order. In many cases, the decision maker is interested in a schedule that has the least variability in job completion times (the most robust schedule). This helps in planning storage and pickup schedules. The variability in completion times can be measured by their range. Let $R\ (P_i)$ $= (U_i - L_i)$ and $R\ (\overline{C})$ be the $\left(\overline{C}_u - \overline{C}_l\right)$ for any given schedule.

## <u>Theorem (3.5)</u>

To minimize the range of the total completion time $\sum C_i$ (or the range of mean completion time $\overline{C}$ ), $R\ (\overline{C})$, jobs should be ordered in non-decreasing order of their processing times ranges $R\ (P_i)$.

**<u>Proof</u>**

The completion time of the first job is

$L_{[1]} \le C_{[1]} \le U_{[1]}$

The completion time of the second job in the sequence S is

$L_{[1]} + L_{[2]} \le C_{[2]} \le U_{[1]} + U_{[2]}$.

The total completion time of the first two jobs is

$2L_{[1]} + L_{[2]} \le C_{[1]} + C_{[2]} \le 2U_{[1]} + U_{[2]}$

Similarly, the total completion time of the first three jobs is

$3L_{[1]} + 2L_{[2]} + L_{[3]} \le C_{[1]} + C_{[2]} + C_{[3]} \le 3U_{[1]} + 2U_{[2]} + U_{[3]}$

In general, for the first k jobs, the total completion time is

$$\sum_{i=1}^{k} (k+1-j)L_{[i]} \le \sum_{i=1}^{k} C_{[i]} \le \sum_{i=1}^{k} (k+1-j)U_{[i]}$$

Therefore, the total completion time of jobs in a sequence S is bounded by the following

$$\sum_{i=1}^{n} (n+1-j)L_{[i]} \le \sum_{i=1}^{n} C_{[i]} \le \sum_{i=1}^{n} (n+1-j)U_{[i]}$$

The range of the total completion times of jos in schedule S is

$$\sum_{i=1}^{n} (n+1-j)\left[U_{[i]} - L_{[i]}\right] = \sum_{i=1}^{n} (n+1-j)R(P_i)$$

The range of $\overline{\mathrm{C}}$ is

$R\left(\overline{C}\right) = \frac{1}{n}\sum_{i=1}^{n} (n+1-j)R(P_i)$, where $\mathrm{R(P_i)}$ is the range of the processing time ( $\mathrm{U_i}$ - $\mathrm{L_i}$ ) which has a fixed and known value for each job.

To minimize the range of $\overline{C}$, $R\ (\overline{C})$, we match the job with the largest range to the position of the smallest coefficient (the last position $n$). i.e. we order jobs in non-decreasing order of their ranges $R\ (P_i)$.

## 3.2 The Three-Point Estimate Approach

In many cases, processing times are unknown. However, experts can provide these estimates: the most likely, the optimistic, and the pessimistic. The most likely (denoted by $m$) is the most realistic estimate of the activity. Statistically, it is the mode. The optimistic estimate (denoted by $a$) is the unlikely but possible time if everything goes well. Statistically, it is the lower bound of the processing time. The pessimistic estimate (denoted by $b$) is the unlikely but possible time if everything goes badly. Statistically, it is the upper bound of the processing time. This approach is used in project scheduling to deal with uncertainty when three point estimates are given in the problem. Processing times are assumed to follow a Beta distribution of *a, b, and m* parameters [71]. The mean of beta distribution is

$$\overline{D} = \frac{a+b+4m}{6}$$

The variance of beta distribution is

$$V = \left(\frac{b-a}{6}\right)^2$$

These values are the expected value and the variance of the processing time that will be found for each activity and then used in the scheduling [70].

In our problem, we have uncertain processing times with bound values. If we are able to obtain the most likely estimate, then we can use this approximation by beta distribution to find the expected value and variance of each processing time, and then we can use it to find the optimum schedule.

In the following theorem, we will prove that, when this three-estimate approach is used to find the expected value and variance of each processing time, then the optimum schedule with respect to expected mean completion time can be found by ordering these expected values in SPT order, the same as in the deterministic case.

## Theorem (3.6)

The optimum schedule with respect to expected mean completion time, when the expected values of the processing times are found by using the three-estimate approach, can be found by ordering these expected values in SPT order.

**Proof**

$$C_1 = P_1 \Rightarrow E(C_1) = E(P_1)$$
$$C_2 = P_1 + P_2 \Rightarrow E(C_2) = E(P_1 + P_2) = E(P_1) + E(P_2)$$
$$C_1 + C_2 = 2P_1 + P_2 \Rightarrow E(C_1 + C_2) = E(2P_1 + P_2) = 2E(P_1) + E(P_2)$$
$$C_n = \sum_{i=1}^{n} P_i \Rightarrow E(C_n) = E\left(\sum_{i=1}^{n} P_i\right) = \sum_{i=1}^{n} E(P_i)$$
$$\sum_{i=1}^{n} C_i = \sum_{i=1}^{n} (n-i+1)P_i \Rightarrow E\left(\sum_{i=1}^{n} C_i\right) = E\left(\sum_{i=1}^{n} (n-i+1)P_i\right) = \sum_{i=1}^{n} (n-i+1)E(P_i)$$

To minimize the expected total completion time (or equivalently the expected mean completion time), we will match the highest coefficient with the smallest expected processing time. That means SPT is giving the minimum expected mean completion time when using the expected processing times.

## 3.3 Heuristic Method to Minimize the Mean Completion Time

For any given sequence, the total completion time can be written as given in (1) as follows:

$$\sum_{i=1}^{n} C_i = \sum_{i=1}^{n} (n-i+1)P_i$$

Processing times are uncertain with known lower and upper bounds. The highest possible value of the total completion times can occur when all processing times have their highest values (at their upper bounds). It is written as:

$$\left(\sum_{i=1}^{n} C_i\right)_U = \sum_{i=1}^{n} (n-i+1)U_i$$

Similarly, the lowest possible value of the total completion time is found when actual processing times are in their lowest values (at their lower bounds) for which the total completion time can be written as:

$$\left(\sum_{i=1}^{n} C_i\right)_L = \sum_{i=1}^{n} (n-i+1)L_i$$

These two values give the range of possible values for the total completion time. However, the total completion time itself is a random variable. For a large number of jobs ($n$), the total completion time can be approximated by a normal distribution (based on the central limit theorem) which is a symmetric distribution with a mean in the middle of its range (variance). Hence, the mean of the total completion time may be approximated by:

$$E\left(\sum_{i=1}^{n} C_i\right) = \frac{\sum_{i=1}^{n}(n-i+1)L_i + \sum_{i=1}^{n}(n-i+1)U_i}{2}$$

$$= \frac{\sum_{i=1}^{n}(n-i+1)(L_i + U_i)}{2}$$

$$= \sum_{i=1}^{n}(n-i+1)P_i^{'}, \qquad \text{where } P_i^{'} = \frac{L_i + U_i}{2}$$

Therefore, for any sequence we expect the total completion time to be as given above.

The scheduling problem now becomes a deterministic one with job processing times equal to $\frac{L+U}{2}$ (which is the middle point in the range of processing times). The optimum schedule for this deterministic problem is known from the literature to be the SPT order.

**<u>Heuristic</u>**

For minimizing the expected total completion time, order jobs in SPT order of $\frac{L_i + U_i}{2}$.

## 3.4 Example

An example will be used to illustrate the use of the above results.

Consider a 5 jobs problem for which the processing times are given in terms of lower and upper bounds.

| J | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| LB | 2 | 1 | 3 | 4 | 6 |
| UB | 4 | 5 | 6 | 5 | 8 |

All theorems in this chapter will be applied to the example.

The necessary data, after some simple calculations, is given in the following table:

| | | | | | |
|---|---|---|---|---|---|
| **Range** | 2 | 4 | 3 | 1 | 2 |
| **(LB + UB)/2** | 3 | 3 | 4.5 | 4.5 | 7 |
| **M** | 3 | 2 | 5 | 4 | 7 |
| **$E(P_i)$** | 3 | 2.33 | 4.83 | 4.17 | 7 |
| **$V(P_i)$** | 0.11 | 0.44 | 0.25 | 0.03 | 0.11 |

Applying theorem 3.1 to the example reduces the solution space from infinite to (5!) sequences. That means we will have 120 sequences.

Applying theorem 3.2 to the example shows that job 1 should always precedes job 4 because ( $U_1 < L_4$ ) in the optimal sequence and jobs 1, 2, 3, and 4 should precede job 5 because ( $U_1 < L_5$ ), ( $U_2 < L_5$ ), ( $U_3 \leq L_5$ ), and ( $U_4 < L_5$ ) in the optimal sequence. In other words, job 5 should be the last job in the optimal sequence.

Applying theorem 3.3 to the example shows that the minimum $\overline{C}_u$ can be found by the sequence 1, 2, 4, 3, 5 or the sequence 1, 4, 2, 3, 5 with a value of $\overline{C}_u = 15$. These sequences result from ordering the jobs in increasing order of their upper bounds.

Applying theorem 3.4 to the example shows that the minimum $\overline{C}_l$ can be found by the sequence 2, 1, 3, 4, 5 with a value of $\overline{C}_l = 7.2$. This sequence results from ordering the jobs in increasing order of their lower bounds.

Applying theorem 3.5 to the example shows that the minimum range of mean completion time $R\ (\overline{C}\,)$ can be found by the sequences 4, 1, 5, 3, 2 or 4, 5, 1, 3, 2 with a value of $R\ (\overline{C}\,) = 5.8$. These sequences result from ordering the jobs in increasing order of their ranges.

Applying theorem 3.6, for minimizing the expected mean completion time, to the example gives the sequence 2, 1, 4, 3, 5 by using the expected processing times with a value of expected mean completion time equal to 4.27.

Applying the heuristic method to the example gives the sequence 2, 1, 4, 3, 5 with a value of mean completion time equal to 4.4.

Applying the complete enumeration method by using the MATLAB program to calculate the range of mean completion times for all possible schedules gives the results in Appendix A. The program is also available in Appendix A.

From the outcome of the program, the following results are found:

- The sequence that gives the minimum mean completion time is [2, 1, 3, 4, 5] with lower mean completion time 7.2.
- The sequences that give the minimum upper value of mean completion time are [1, 2, 4, 3, 5] and [1, 4, 2, 3, 5] with upper mean completion time 15.
- The minimum range of mean completion time is 5.8, and the sequences that give this range are [4, 1, 5, 3, 2] and [4, 5, 1, 3, 2].

These results confirm the results obtained by the characterization of the optimum solution and the heuristic method.

# CHAPTER 4

# Total Earliness and Tardiness

In this chapter, the objective is to minimize earliness and tardiness with a common due date known in advance. This means that we want all jobs to finish as close as possible to the due date. The importance of this objective comes from the just-in-time (JIT) system. In JIT, we want all jobs to finish at the due date, to avoid holding and storing costs if the job finishes early, and to save penalties and goodwill costs resulting from late finishing. Objective functions that include both earliness and tardiness are known as non-regular performance measures. When all data are known in advance (deterministic), certain properties are identified, and constructive solutions are developed for some special cases [27]. Expected values are used to measure schedule performance under some types of uncertainties that can be described as random variables with known expected values [27, 28]. However, when the uncertainty cannot be described by a distribution function or by the expected values, the problem becomes untraceable for the expected values. Different sets of objective functions may translate the need of the scheduler in such cases.

Early-Tardy problems in the literature are classified differently according to the due date. When the due date is a decision variable, the problem becomes due-date determination plus optimum scheduling. Usually, this problem is easier to handle, since we have an additional degree of freedom in solving the problem. This class of problems is classified as "unconstrained". When the decision maker is constrained by a given due date, the problem becomes a "constrained" scheduling problem. If the due date is large (compared to processing times) it does not restrict the choice of feasible schedules given to the scheduler, and the problem is then called "unrestrictive". However, when the due date is fixed for a certain "small value", it becomes "restrictive" with limited choices of possible schedules. The latter is usually more difficult to handle. The unconstrained problem is unrestrictive by nature. A due date is restrictive "small" if its value is less than the optimum due date of the unconstrained problem. The unconstrained and the unrestrictive sub-versions of the deterministic version of the problem are polynomially solvable, whereas the restrictive version of the problem is known to be NP- hard. The solution of the deterministic version of the problem is known to have a V-shape in which early jobs should be sequenced in LPT order and tardy jobs in SPT order [5].

## 4.1 Assumptions and Notations

For this objective function, some additional assumptions and notations should be mentioned before any further work in this chapter.

- There are *n* jobs, with uncertain processing times (for each processing time $P_i$ there is an upper bound $U_i$ and a lower bound $L_i$ known in advance).

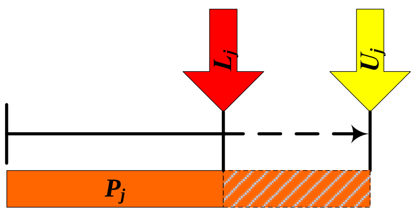



Figure 4-1: Lower and upper bounds of bounded processing time.

- All jobs have a common due date *d*.

- The optimality criterion is to minimize the summation of earliness and tardiness, that is $\sum_{i=1}^{n} |C_i - d|$, where $C_i$ is the completion time of job $i$.
- This problem can be classified as $1/L_i \le P_i \le U_i(uncertain)/\sum_{i=1}^{n}|C_i - d|$, using the notation used by Pinedo [26].
- In any sequence $S$, three subsequences are identified:

  $S_e$ : the subsequence of jobs that are definitely early (early with probability 1).

  $$S_e = \{\, j=1, 2, \ldots, b, \qquad b = \max_k \left( \sum_{j=1}^{k} U_j \le d \right) \}$$

  $S_t$ : the subsequence of jobs that are definitely tardy (late with probability 1).

  $$S_t = \{\, j=n-a+1, n-a+2, \ldots, n, \quad a = n - \max_k \left( \sum_{j=1}^{k} L_j \le d \right) \}$$

  $S_x$ : the subsequence of the remaining jobs that can go both ways, each job can be early or tardy depending on its actual processing times and the jobs preceding it.

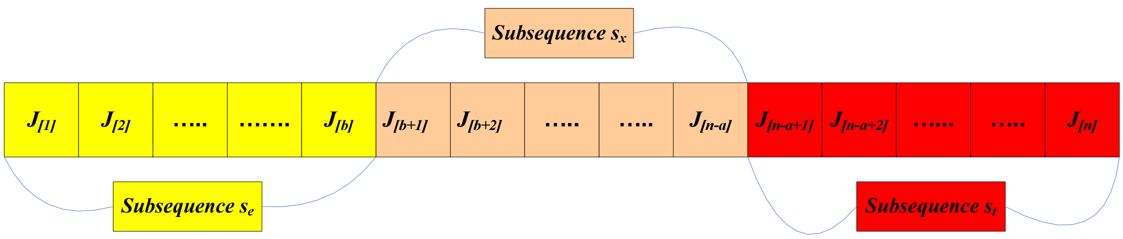



Figure 4-2: The three subsequences of any sequence S.

- The number of jobs in the subsequence $S_e$ is $b$.
- The number of jobs in the subsequence $S_t$ is $a$.
- The number of jobs in the subsequence $S_x$ is $n$-$a$-$b$.

## 4.2 Formulation

The objective function can be written in terms of processing times, and this will make it easier to deal with. This objective function is different from the mean completion time, because it contains two parts. The first part is for earliness, whereas the second is for tardiness. This requires dealing with each part separately. Thus, we will find the objective function for early jobs (earliness) and then for tardy jobs (tardiness).

The contribution of early jobs (jobs in the subsequence $S_e$) to the objective value is:

$$\sum_{j=1}^{b}\left(d - C_{[j]}\right) = \sum_{j=1}^{b}\sum_{h=j+1}^{b} P_{[h]} = bx_1 + \sum_{j=1}^{b}(b-j)P_{[j]}$$

$$\begin{aligned}
d - C_{[b]} &= x_1 \\
d - C_{[b-1]} &= x_1 + P_{[b]} \\
d - C_{[b-2]} &= x_1 + P_{[b]} + P_{[b-1]} \\
&\;\;. \\
&\;\;. \\
d - C_{[1]} &= x_1 + P_{[b]} + P_{[b-2]} + P_{[b-3]} + \dots + P_{[2]}
\end{aligned}$$

$$\begin{aligned}
\sum_{j=1}^{b}\left(d - C_{[j]}\right) &= bx_1 + (b-1)P_{[b]} + (b-2)P_{[b-1]} + \dots.. \\
&= bx_1 + \sum_{j=1}^{b}(j-1)P_{[j]}
\end{aligned}$$

After some simplifications, we will have:

$$\sum_{j=1}^{b}\left(d - C_{[j]}\right) = \sum_{j=1}^{b}(j-1)P_{[j]} + bx_1 \qquad (2)$$

Where $x_1$ is a random variable that represents the processing times of all jobs in $S_x$ finishing before the due date.

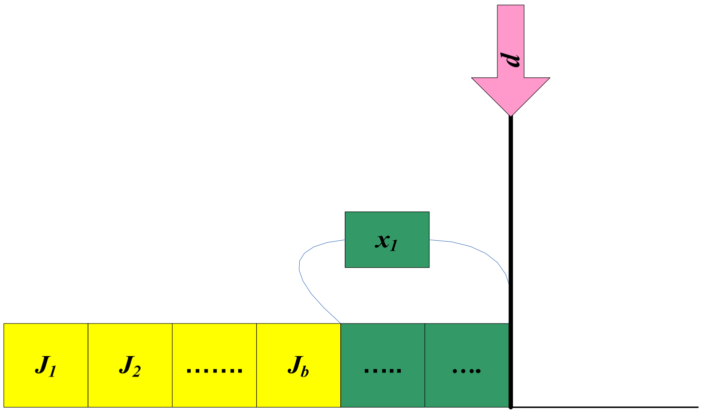



Figure 4-3: The early set and the due date.

For tardy jobs (jobs in the subsequence $S_t$), we have:

$$\sum_{j=1}^{a} \left| C_{[n-j+1]} - d \right| = \sum_{j=1}^{a} (a - j + 1) P_{[j]} + a x_2 \,,$$

where $x_2$ is a random variable that represents the processing times of all jobs in $S_x$ finishing after the due date.

After some simplifications, we will have:

$$\sum_{j=1}^{a} \left| C_{[n-j+1]} - d \right| = \sum_{j=1}^{a} (j) P_{[n-j+1]} + a x_2 \qquad (3)$$

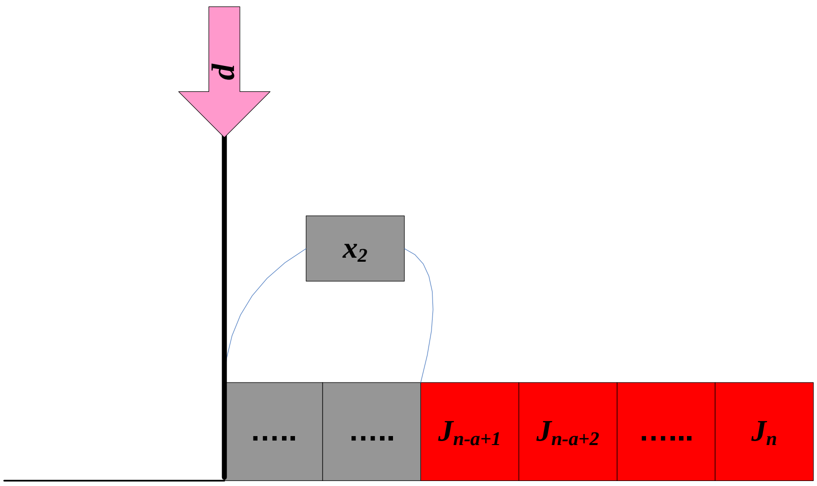



Figure 4-4: The tardy set and the due date.

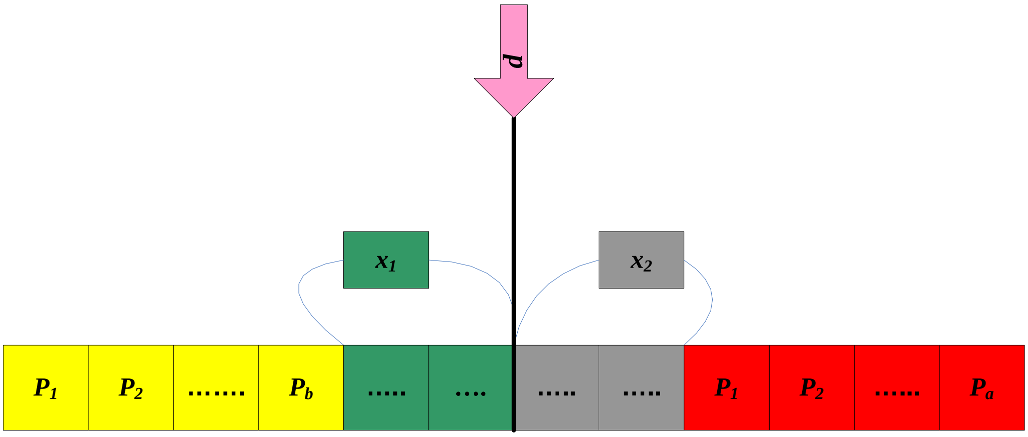



Figure 4-5: The complete sequence and the due date.

After finding the expression for both early and tardy jobs in terms of processing times, we will write the objective function completely in terms of processing times. Since we deal with uncertain processing times and we have three subsequences, we will assume that the subsequence $S_x$ gives $Z$ as an objective value (earliness + tardiness) for all jobs in $S_x$.

The objective function in terms of processing times will be as follows:

$$\sum_{j=1}^{n} \left| C_j - d \right| = \sum_{j=1}^{b} (j-1) P_{[j]} + \sum_{j=1}^{a} (j) P_{[n-j+1]} + a x_2 + b x_1 + Z \tag{4}$$

where $Z$ = the objective values (earliness or tardiness) of jobs that are in the subsequence $S_x$ (jobs in positions $b+1$, $b+2 ... n-a$).

The only available information in this problem is the lower and upper bounds of the uncertain processing times, and the objective function is to minimize earliness and tardiness. Next, we will consider the case when the due date is known and the case when it is considered to be a decision variable.

## 4.3 Constrained Scheduling

In this section, the due date is given (known in advance) and we need to determine the optimal schedule that will minimize total earliness and total tardiness with these uncertain processing times. The characteristics of the optimal solution will be determined.

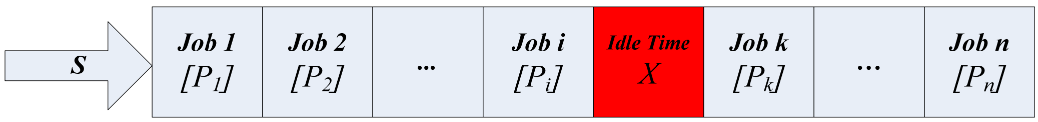
The following theorem proves that, in this problem, there is no idle time inserted between jobs.

**<u>Theorem (4.1)</u>**

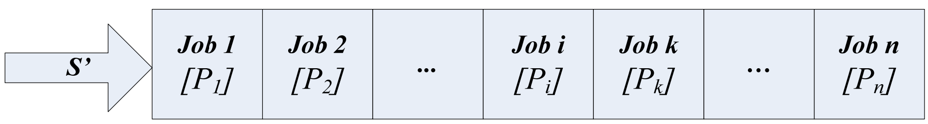
In this problem, $1\,/\,(\text{uncertain processing times})\ L_i \le P_i \le U_i\,/\sum_{i=1}^{n}\left|C_{[i]}-d\right|$, there is an optimal solution in which no idle time is inserted between jobs.

**<u>Proof</u>**

Let us assume that $S$ is an optimal schedule with an idle time of length $X$ inserted between two jobs as follows:




And let $S'$ be the resulting schedule after removing the idle time from $S$ as follows:




Clearly $C_i' \le C_i$ for all $i = 1, 2, ..., n$, where $C_i$ and $C_i'$ are completion times of jobs in $S$ and $S'$ respectively.

$$C_j^{'} = C_j - X \qquad \text{where } X \text{ is constant}$$
$$= \sum_{i=1}^{j} P_i - X \qquad \text{where} \sum_{i=1}^{j} P_i \text{ is uncertain}$$

$$C_j = C_j^{'} \qquad \text{for all } j = 1,2,...,i$$
$$C_j \geq C_j^{'} \qquad \text{for all } j = k, k+1,...,n$$

The completion times for all jobs preceding job *k* remain unchanged, $C_j = C_j^{'}$ while the completion times for all jobs after job *k* (including job *k*) are smaller by *X*, $C_j \geq C_j^{'}$. This is true regardless of the fact that $P_i$s are uncertain, since the actual processing times are independent of the actual starting time of processing.

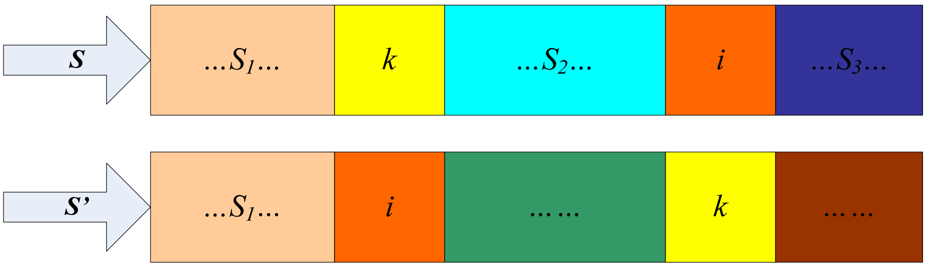
In the following theorem, we will define one of the dominance relations in this objective.

## Theorem (4.2)

For two jobs *i* and *k* in the early set $S_e$, there is an optimal sequence in which job *i* precedes job *k* if $U_k \leq L_i$.

**Proof**

Assume that the sequence $S=(s_1, k, s_2, i, s_3)$ is the optimal sequence for early jobs with $U_k \leq L_i$ and *job k* in position *x* and *job i* in position *y*, where $x < y$ and the sequence *S'* $=(s_1, i, s_2, k, s_3)$ is another sequence that results from interchanging jobs *i* and *k*.




From (3):

Objective function for the sequence *S'* is:

$$\begin{aligned} f(S') = \sum_{j=1}^{n}\left|C_j - d\right| &= \sum_{j=1}^{b}(j-1)P_{[j]} + \sum_{j=1}^{a}(j)P_{[n-j+1]} + Z + bx_1 + ax_2 \\ &= 0P_{[1]} + 1P_{[2]} + 2P_{[3]} + \ldots.. + (x-1)P_i + xP_{[x+1]} + \ldots.. + (y-1)P_k \\ &+ yP_{[y+1]} + \ldots.. + (b-1)P_{[b]} + \sum_{j=1}^{a}(j)P_{[n-j+1]} + Z + bx_1 + ax_2 \end{aligned}$$

Objective function for the sequence *S* is:

$$\begin{aligned} f(S) = \sum_{j=1}^{n}\left|C_j - d\right| &= \sum_{j=1}^{b}(j-1)P_{[j]} + \sum_{j=1}^{a}(j)P_{[n-j+1]} + Z + bx_1 + ax_2 \\ &= 0P_{[1]} + 1P_{[2]} + 2P_{[3]} + \ldots.. + (x-1)P_k + xP_{[x+1]} + \ldots.. + (y-1)P_i \\ &+ yP_{[y+1]} + \ldots.. + (b-1)P_{[b]} + \sum_{j=1}^{a}(j)P_{[n-j+1]} + Z + bx_1 + ax_2 \end{aligned}$$

The difference between the two objective functions is:

$$\begin{aligned} f(S) - f(S') &= [(x-1)P_k + (y-1)P_i] - [(x-1)P_i + (y-1)P_k] \\ &= (x-y)P_k + (y-x)P_i \\ &= (y-x)(P_i - P_k) \geq 0 \qquad \text{( positive with probability 1)} \end{aligned}$$

Since $y > x$ and $P_i \geq P_k$ with probability 1 (since $U_k \leq L_i$).

Since *S* is optimal, then *S'* is also optimal.

After proving this theorem, we can generalize this result so that, if we have a job that is larger than all other jobs, then it should be in the first position as shown in the following lemma.

## Lemma (4.1)

If there is a job $j$ having $L_j \geq \max_{i \neq j}(U_i)$, then there is an optimal sequence in which job $j$ is in the first position of the optimal sequence.

**Proof**

From the result of theorem (4.2), the coefficient of the first job in the objective function is zero, and this job has the largest processing time with probability 1.

Another result from the previous theorem is that, if we have a job that is smaller than all other jobs (with probability 1), then it should take the position with the largest multiplier, namely the last early job in the subsequence $S_e$ as shown in the following lemma.

## Lemma (4.2)

If there is a job $j$ having $U_j \leq \min_{i \neq j}(L_i)$, then there is an optimal sequence in which job $j$ is on the due date (the last position of the set of early jobs).

**Proof**

From the result of theorem (4.2), the coefficient of the last job in the set of early jobs is the largest, and this job has the smallest processing time with probability 1.

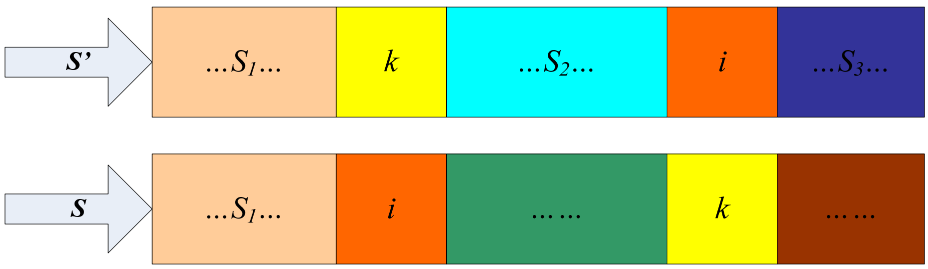
The previous theorem showed the dominance relation for early jobs. The following theorem introduces a similar dominance relation for tardy jobs. So, if we have two jobs that are tardy with probability 1, in which the lower bound of one job is greater than or equal to the upper bound of the other job, then the smaller job must precede the larger in any optimal sequence for tardy jobs.

## Theorem (4.3)

For two jobs $i$ and $k$ that are definitely in the tardy set $S_t$, there is an optimal sequence in which job $k$ precedes job $i$ if $U_k \leq L_i$.

### Proof

Assume that $S = (s_1, i, s_2, k, s_3)$ is the optimal sequence for tardy jobs in which $U_k \leq L_i$ and *job* $i$ in position $x$ and *job* $k$ in position $y$, where $x < y$, and the sequence $S'$ $= (s_1, k, s_2, i, s_3)$ is another sequence that results from interchanging jobs $i$ and $k$ and $U_k < L_i$.



From (3):

The objective function for the sequence *S* is:

$$f(S)=\sum_{j=1}^{n}\left|C_j-d\right|=\sum_{j=1}^{b}(j-1)P_{[j]}+\sum_{j=1}^{a}(j)P_{[n-j+1]}+Z+bx_1+ax_2$$
$$=\sum_{j=1}^{b}(j-1)P_{[j]}+Z+bx_1+ax_2+(a)P_{[n-a+1]}+.....+(a-x)P_i$$
$$+....+(a-y)P_k+.....+P_{[n]}$$

The objective function for the sequence *S'* is:

$$f(S')=\sum_{j=1}^{n}\left|C_j-d\right|=\sum_{j=1}^{b}(j-1)P_{[j]}+\sum_{j=1}^{a}(j)P_{[n-j+1]}+Z+bx_1+ax_2$$
$$=\sum_{j=1}^{b}(j-1)P_{[j]}+Z+bx_1+ax_2+(a)P_{[n-a+1]}+.....+(a-x)P_k$$
$$+....+(a-y)P_i+.....+P_{[n]}$$

After that, we find:

$$f(S)-f(S') = [(a-x)P_i+(a-y)P_k] - [(a-x)P_k+(a-y)P_i]$$

$$=(x-y)P_k + (y-x)P_i$$

$$=(y-x)(P_i - P_k) \geq 0 \quad \text{( positive with probability 1)}$$

Since $y > x$ and $P_i \geq P_k$ with probability 1 (since $U_k \leq L_i$).

Since *S* is optimal, then *S'* is also optimal.

### 4.3.1 Ranges of Earliness and Tardiness

Minimizing the earliness is a special case of the early-tardy problem. Here, we investigate the earliness-related objective functions under uncertainty. One method is to consider the range of earliness by using the ranges of processing times, and this is what is done in the following theorem. The earliness performance measure is related to the in-process inventory, and hence reducing the average earliness is directly related to minimizing the average in-process inventory. Minimizing its range $R\ (\overline{E})$ will help in controlling the inventory space and better planning decisions for the inventory level.

## Theorem (4.4)

To minimize the range of the total earliness, $R\ (\overline{E})$, jobs should be ordered in decreasing order of their ranges $R\ (P_i)$.

### Proof

There are b jobs in the set of early jobs $S_e$ (jobs completing before $d$ with probability 1)

$E_i = \max(0, d - C_i)$

$$d - L_{[1]} \leq E_{[1]} \leq d - U_{[1]}$$
$$d - \left(L_{[1]} + L_{[2]}\right) \leq E_{[2]} \leq d - \left(U_{[1]} + U_{[2]}\right)$$
$$d - \left(L_{[1]} + L_{[2]} + L_{[3]}\right) \leq E_{[3]} \leq d - \left(U_{[1]} + U_{[2]} + U_{[3]}\right)$$

.

.

The total earliness is therefore limited by

$$bd - b\left(L_{[1]}\right) - (b-1)\left(L_{[2]}\right) - \ldots \leq \sum_{i=1}^{b} E \leq bd - b\left(U_{[1]}\right) - (b-1)\left(U_{[2]}\right) - (b-2)\left(U_{[3]}\right) - \ldots$$

The range of the total earliness of jobs in $S_e$ is

$$R\left(\sum E\right) = \left\{bd - b\left(U_{[1]}\right) - (b-1)\left(U_{[2]}\right) - (b-2)\left(U_{[3]}\right) - \ldots\right\} - \left\{bd - b\left(L_{[1]}\right) - (b-1)\left(L_{[2]}\right) - \ldots\right\}$$
$$= -b\left(U_{[1]} - L_{[1]}\right) - (b-1)\left(U_{[2]} - L_{[2]}\right) - (b-2)\left(U_{[3]} - L_{[3]}\right) - \ldots\ldots$$
$$R\left(\sum E\right) = \sum_{i=1}^{b} -(b-i+1)\left[U_{[i]} - L_{[i]}\right] = \sum_{i=1}^{n} -(b-i+1)R\left(P_i\right)$$

To minimize the range $R\ (\overline{E})$, we assign the job with the largest range to the position of the highest coefficient (the first position), i.e. we order jobs in decreasing order of their ranges $R\ (P_i)$.

Next, we need to consider tardiness alone, and this can be done by using the same technique, namely minimizing the range of total tardiness by using the ranges of processing times, and this is what is the following theorem is about. The tardiness performance measure is related to the inventory and storage after the production, and hence reducing the average tardiness is directly related to minimizing the average inventory. Minimizing its range $R\ (\overline{T}\,)$ will help in controlling the inventory space and in achieving better planning decisions for inventory and storage levels.

## Theorem (4.5)

To minimize the range of the total tardiness, $R\ (\overline{T}\,)$, jobs should be ordered in increasing order of their ranges $R\ (P_i)$.

**Proof**

there are $a$ jobs in the set of tardy jobs $\mathrm{S_t}$ (jobs completing after $d$ with probability 1)

$$T_i = \max(0, C_i - d)$$

$$L_{[1]} - d \le T_{[1]} \le U_{[1]} - d$$
$$\left(L_{[1]} + L_{[2]}\right) - d \le T_{[2]} \le \left(U_{[1]} + U_{[2]}\right) - d$$
$$\left(L_{[1]} + L_{[2]} + L_{[3]}\right) - d \le T_{[3]} \le \left(U_{[1]} + U_{[2]} + U_{[3]}\right) - d$$

.

.

The total tardiness is therefore limited by

$$-ad + b\left(L_{[1]}\right) + (a-1)\left(L_{[2]}\right) + \ldots \le \sum_{i=1}^{a} T \le -ad + a\left(U_{[1]}\right) + (a-1)\left(U_{[2]}\right) + (a-2)\left(U_{[3]}\right) + \ldots$$

The range of total tardiness of jobs in $S_t$ is

$$R\left(\sum T\right) = \left\{-ad + a\left(U_{[1]}\right) + (a-1)\left(U_{[2]}\right) + (a-2)\left(U_{[3]}\right) + \ldots\right\} - \left\{-ad + b\left(L_{[1]}\right) + (a-1)\left(L_{[2]}\right) + \ldots\right\}$$
$$= a\left(U_{[1]} - L_{[1]}\right) + (a-1)\left(U_{[2]} - L_{[2]}\right) + (a-2)\left(U_{[3]} - L_{[3]}\right) + \ldots..$$
$$R\left(\sum T\right) = \sum_{i=1}^{b} (a-i+1)\left[U_{[i]} - L_{[i]}\right] = \sum_{i=1}^{n} (a-i+1) R\left(P_i\right)$$

To minimize the range $R(\overline{T})$, we assign the job with the largest range to the position of the smallest coefficient (the last position $n$), i.e. we order jobs in non-decreasing order of their ranges $R(P_i)$.

## 4.4 Common Due-Date Determination

In this section, a sequence is given (known in advance) and we need to find the common due date that will minimize the total earliness and total tardiness under uncertainty of processing times.

The following theorem proves that, if the sequence is given, then the optimum due date will be between the completion time of the middle job (in position $n/2$) and the completion time of the following job (in position $[n/2]+1$). This is similar to the deterministic case.

**Theorem (4.6)**

If the sequence is given, then the optimum due date can be anywhere between the completion time of the job in position $n/2$ and the completion time of the job in position $[n/2]+1$.

**Proof**

Let us assume that there are $b$ early jobs and $n-b$ tardy jobs. Then, the objective function will be:

$$\sum_{i=1}^{n}\left|C_i - d\right| = (d - C_1) + (d - C_2) + \ldots + (d - C_b) + (C_{b+1} - d) + (C_{b+2} - d) + \ldots + (C_n - d)$$

$$= bd - (n-b)d - \sum_{i=1}^{b} C_i + \sum_{i=b+1}^{n} C_i$$

$$= d(2b - n) - \sum_{i=1}^{b} C_i + \sum_{i=b+1}^{n} C_i$$

After that, we need to find the value of $d$ that will minimize the objective function. So, we will take the first derivative with respect to $d$ for the objective function and equate it to zero.

$$\min_d \sum_{i=1}^{n}\left|C_i - d\right| = d(2b-n) - \sum_{i=1}^{b} C_i + \sum_{i=b+1}^{n} C_i$$

$$(2b - n) = 0 \Rightarrow b = \frac{n}{2}$$

$$\text{if } n \text{ is odd, then } b = \frac{n}{2} \pm \frac{1}{2} = \frac{n+1}{2}$$

## 4.5 Unconstrained

In this section, there is no information about the due date and the sequence. We need to find the optimum sequence as well as the optimum due date.

In the following theorem, the general due-date range is found. This means that for all sequences the due date should be within this range. In this theorem, the first step is to find a sequence, and from the sequence we can find the due date (by theorem 4.6).

Processing times are not known exactly, and we are given only the lower and upper bounds. If we separate the problem into two deterministic problems, one with lower bounds, and one with upper bounds then we can generate the optimal sequence for each problem, and from that sequence we can find the due date (assuming deterministic due date determination). If we find the due dates for the best-case scenario (lower bounds) and the worst-case scenario (upper bounds) then the range between these two values will give us the range of the due date.

**Theorem (4.7)**

The general due-date range for any problem is the range produced by finding the due date by using only lower bounds as processing times and then by using only upper bounds. In each problem (using lower and upper bounds), the optimal sequence can be found from the V-shape theorem. After that, the due date will be found by using theorem 4.6. Finally, the due-date range will be the merge of these two due-date ranges.

## 4.6 The Three-Point Estimate Approach

In many cases, processing times are unknown. However, the experts can provide these estimates: the most likely, the optimistic, and the pessimistic. The most likely (denoted by $m$) is the most realistic estimate of the activity. Statistically, it is the mode. The optimistic estimate (denoted by $a$) is the unlikely but possible time if everything goes well. Statistically, it is the lower bound of the activity's time. The pessimistic estimate (denoted by $b$) is the unlikely but possible time if everything goes badly. Statistically, it is the upper bound of the activity's time. This approach is used in project scheduling to deal with uncertainty when three time-estimates are given in the problem. Processing times are assumed to follow a Beta distribution of *a, b, and m* parameters [71]. The mean of beta distribution is

$$\overline{D} = \frac{a + b + 4m}{6}$$

The variance of beta distribution is

$$V = \left( \frac{b - a}{6} \right)^2$$

These values are the expected value of the activity time and the variance, and they will be found for each activity and then used in the scheduling [70].

In our problem, we have uncertain processing times with bound values. If we are able to obtain the most likely estimate, then we can use this approximation by beta distribution to find the expected value and variance of each processing time, and then use it to find the optimum schedule.

In the following theorem, we will prove that, when this three-estimate approach is used to find the expected value and variance of each processing time, then the optimum schedule with respect to total earliness and tardiness time can be found by the V-shape schedule, the same as in the deterministic case. That means early jobs should be sequenced in LPT order and tardy jobs should be ordered in SPT.

## Theorem (4.8)

The optimum schedule with respect to expected total earliness and tardiness, when the expected values of processing times are found by using the three-estimate approach, can be found by the V-shape schedule. That means early jobs should be sequenced in LPT order and tardy jobs should be ordered in SPT.

**Proof**

$$C_1 = P_1 \Rightarrow E(C_1) = E(P_1)$$
$$C_2 = P_1 + P_2 \Rightarrow E(C_2) = E(P_1 + P_2) = E(P_1) + E(P_2)$$
$$C_1 + C_2 = 2P_1 + P_2 \Rightarrow E(C_1 + C_2) = E(2P_1 + P_2) = 2E(P_1) + E(P_2)$$
$$C_n = \sum_{i=1}^{n} P_i \Rightarrow E(C_n) = E\left(\sum_{i=1}^{n} P_i\right) = \sum_{i=1}^{n} E(P_i)$$
$$\sum_{j=1}^{n} \left|C_j - d\right| = \sum_{j=1}^{b} (j-1)P_{[j]} + \sum_{j=1}^{a} (j)P_{[n-j+1]}$$
$$\Rightarrow E\left(\sum_{j=1}^{n} \left|C_j - d\right|\right) = E\left(\sum_{j=1}^{b} (j-1)P_{[j]} + \sum_{j=1}^{a} (j)P_{[n-j+1]}\right) = \sum_{j=1}^{b} (j-1)E\left(P_{[j]}\right) + \sum_{j=1}^{a} (j)E\left(P_{[n-j+1]}\right)$$

To minimize the expected total earliness and tardiness, we will match the highest coefficient with the smallest expected processing time. That means LPT order is used in early jobs and SPT is used with tardy jobs when using expected processing times.

## 4.7 Example

| J | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| **LB** | 2 | 1 | 3 | 4 | 6 |
| **UB** | 4 | 5 | 6 | 5 | 8 |
| **Due Date** | 14 | 14 | 14 | 14 | 14 |

All theorems in this chapter will be applied to the example.

The necessary data, after some simple calculations, is given in the following table:

| | | | | | |
|---|---|---|---|---|---|
| **Range** | 2 | 4 | 3 | 1 | 2 |
| **m** | 3 | 2 | 5 | 4 | 7 |
| **E($P_i$)** | 3 | 2.33 | 4.83 | 4.17 | 7 |
| **V($P_i$)** | 0.11 | 0.44 | 0.25 | 0.03 | 0.11 |

Applying theorem 4.1 to the example reduces the solution space from infinite to (5!) sequences. That means we will have 120 sequences.

Applying theorem 4.2 to the example shows that, if jobs 1 and 4 are in the early set $S_e$, job 4 should always precede job 1, because ( $U_1 < L_4$ ) in the optimal sequence, and, if all jobs are in the early set $S_e$, then job 5 should precede jobs 1, 2, 3, and 4, because ( $U_1 < L_5$ ), ( $U_2 < L_5$ ), ( $U_3 \leq L_5$ ), and ( $U_4 < L_5$ ) in the optimal sequence. In other words, job 5 should be the first job in the optimal sequence.

Applying theorem 4.3 to the example shows that, if jobs 1 and 4 are in the Tardy set $S_t$, then job 1 should always precedes job 4, because ( $U_1 < L_4$ ) in the optimal sequence, and, if all jobs are in the tardy set $S_t$, then jobs 1, 2, 3, and 4 should precede job 5, because ( $U_1 < L_5$ ), ( $U_2 < L_5$ ), ( $U_3 \leq L_5$ ), and ( $U_4 < L_5$ ) in the optimal sequence. In other words, job 5 should be the last job in the optimal sequence.

Applying lemma 4.1 to the example shows that job 5 should be in the first position in the optima sequence, because its lower bound is greater than or equal to the upper bounds of all other jobs.

Lemma 4.2 cannot be applied, because no job satisfies the condition.

Theorems 4.4 and 4.5 will not be applied, because in this example we are interested in earliness and tardiness together, not separately.

Theorem 4.6 cannot be applied because, there is no specified or given sequence.

Applying theorem 4.7 to the example shows that, when using the lower bounds, the sequence is 5, 3, 2, 1, 4 and the range of the due date is from 10 to 12, but, when using the upper bound, the sequence is 5, 2, 1, 4, 3 and the range is from 17 to 22. So the range of the due date is from 10 to 22.

Applying theorem 4.8 to the example gives the following sequence 5, 4, 2, 1, 3 by using expected processing times with a value of expected total earliness and tardiness equal to 20.16.

Applying the complete enumeration method, by using the MATLAB program for the calculation of the total earliness and tardiness for all possible schedules, gives the results in Appendix B. The program is also available in Appendix B.

From the outcome of the program, the following results are found:

- The minimum total earliness and tardiness with respect to lower bounds is 15, and the sequence that gives this value is [5, 4, 3, 2, 1].
- The minimum total earliness and tardiness with respect to upper bounds is 32, and the sequences that give this value are [3,4,1,2,5], [5,2,1,4,3], and [5,4,1,2,3].

These results confirm the results obtained by the characterization of the optimum solution and the heuristic method.

# CHAPTER 5

# Conclusion

The problem considered in this thesis is the scheduling of a certain number of jobs on a single machine so that a certain objective function is minimized. This problem is the most widely standard problem in scheduling theory, as it represents a building block for more complicated scheduling. This problem is usually studied under the assumption of the availability of accurate information related to processing times and all other parameters of the problem. This assumption is not always true in reality.

Scheduling under uncertainty is a common part of production planning in various industries. However, the scheduling literature seldom deals with uncertainty because it is complex and difficult to analyze. Even when uncertainty is considered, it is assumed to have a certain probability distribution which is, in practice, difficult to know. Experts in the production system can usually estimate the most likely, the pessimistic and the optimistic estimates of needed information. Making scheduling decisions based on these three estimates for the activity processing time is the subject of this thesis.

Two families of objective functions are usually considered in scheduling theory: objectives related to job completion times, and objectives related to meeting due dates. Both types are considered in this thesis for measuring the performance of the schedule.

In this thesis, the first objective is to minimize the mean completion time. By this minimization, the in-process inventory will be minimized. The optimal schedule, when the environment is a single machine, should have no idle time inserted. After that, a new dominance relation is presented and proved. Then, the characterization of the optimum solution that will reduce the upper bound of mean completion time is proved. Moreover, another characterization that will reduce the lower bound is shown and proved. Another related performance measure is the range of mean completion time, and this performance measure is minimized by a given method. The three-point estimate approach is used to minimize the expected mean completion time, when the third-point estimate is known, and a heuristic is presented to give a good solution to this problem.

The second objective is to minimize the total earliness and tardiness. This objective is used to apply the JIT system. In constrained scheduling, where the due date is known, the optimum schedule will have three subsequences. The first one is for early jobs, the second one is for tardy jobs, and the last one is for jobs that are in between and can go both ways. In this optimum schedule, no idle time will be inserted, and this is proved. A dominance relation is proved for early jobs and another dominance relation is proved for tardy jobs. After that, the ranges of total earliness and tardiness are minimized respectively. Then, a method to find the optimum due date is presented. However, in unconstrained scheduling, the only result that can be achieved is the range of due date. Finally, the three-point estimate approach is used to minimize the expected total earliness and tardiness, and a heuristic is presented to find a good solution to this problem.

Complete enumeration for a specific 5-jobs example was done by using MATLAB to confirm our theoretical results, and it was confirmed.

## 5. 1 Limitations and Future Directions

There are several limitations to this research, related to the structure of the problem and its data. These limitations usually exist in most problems in scheduling theory.

Some of the limitations are:

1. The available points of estimation of job processing time are independent.
2. Job processing times are independent from each other.

Several directions for future research can be considered. Some of these are:

1. Different production environments can be studied, such as parallel, flow shop, job shop, and open shop under uncertainty in processing times.
2. Jobs can have sequence-dependent setup times or have some constrained orders.
3. Planned maintenance and machine breakdown can be included in this problem.

# Appendix A

**Solution Method for the Mean Completion Time using MATLAB**

Another solution method is by complete enumeration (finding all feasible schedules) and then choosing the schedule that gives the minimum objective value. MATLAB was used to find all possible sequences by considering the processing times' estimates (lower and upper bounds). The program that was used is:

```
function [S1, S2, C1, C2, TCT1, TCT2, MCT1, MCT2] = sch1(A)
numofent = size(A, 2);
S1 = perms(A(1, :));
for i = 1:factorial(numofent)
   C1(i, 1) = S1(i, 1);
   for j = 2:numofent
      C1(i, j) = S1(i, j) + C1(i, j-1);
   end
   TCT1(i, 1) = sum(C1(i, :));
   MCT1(i, 1) = mean(C1(i, :));
end
S2 = perms(A(2, :));
for i = 1:factorial(numofent)
   C2(i, 1) = S2(i, 1);
   for j = 2:numofent
      C2(i, j) = S2(i, j) + C2(i, j-1);
   end
   TCT2(i, 1) = sum(C2(i, :));
   MCT2(i, 1) = mean(C2(i, :));
end
```

To illustrate the use of this program, all possible schedules are generated and evaluated for the following problem (which was discussed in chapter 3):

| J | 1 | 2 | 3 | 4 | 5 | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| LB | 2 | 1 | 3 | 4 | 6 | | | | | | | | | |
| UB | 4 | 5 | 6 | 5 | 8 | | | | | | | | | |
| Range | 2 | 4 | 3 | 1 | 2 | | | | | | | | | |
| TCT is total completion time | | | | | | | MCT is mean completion time | | | | | | | |
| i | Bound | Sequence of Jobs | | | | | | | | | | | | |
| | | [1] | [2] | [3] | [4] | [5] | $C_{[1]}$ | $C_{[2]}$ | $C_{[3]}$ | $C_{[4]}$ | $C_{[5]}$ | TCT | MCT | Range |
| 1 | L | 1 | 2 | 3 | 4 | 5 | 2 | 3 | 6 | 10 | 16 | 37 | 7.4 | 7.8 |
| | U | | | | | | 4 | 9 | 15 | 20 | 28 | 76 | 15.2 | |
| 2 | L | 1 | 2 | 3 | 5 | 4 | 2 | 3 | 6 | 12 | 16 | 39 | 7.8 | 8 |
| | U | | | | | | 4 | 9 | 15 | 23 | 28 | 79 | 15.8 | |
| 3 | L | 1 | 2 | 4 | 3 | 5 | 2 | 3 | 7 | 10 | 16 | 38 | 7.6 | 7.4 |
| | U | | | | | | 4 | 9 | 14 | 20 | 28 | 75 | 15 | |
| 4 | L | 1 | 2 | 4 | 5 | 3 | 2 | 3 | 7 | 13 | 16 | 41 | 8.2 | 7.2 |
| | U | | | | | | 4 | 9 | 14 | 22 | 28 | 77 | 15.4 | |
| 5 | L | 1 | 2 | 5 | 3 | 4 | 2 | 3 | 9 | 12 | 16 | 42 | 8.4 | 7.8 |
| | U | | | | | | 4 | 9 | 17 | 23 | 28 | 81 | 16.2 | |
| 6 | L | 1 | 2 | 5 | 4 | 3 | 2 | 3 | 9 | 13 | 16 | 43 | 8.6 | 7.4 |
| | U | | | | | | 4 | 9 | 17 | 22 | 28 | 80 | 16 | |
| 7 | L | 1 | 3 | 2 | 4 | 5 | 2 | 5 | 6 | 10 | 16 | 39 | 7.8 | 7.6 |
| | U | | | | | | 4 | 10 | 15 | 20 | 28 | 77 | 15.4 | |
| 8 | L | 1 | 3 | 2 | 5 | 4 | 2 | 5 | 6 | 12 | 16 | 41 | 8.2 | 7.8 |
| | U | | | | | | 4 | 10 | 15 | 23 | 28 | 80 | 16 | |
| 9 | L | 1 | 3 | 4 | 2 | 5 | 2 | 5 | 9 | 10 | 16 | 42 | 8.4 | 7 |
| | U | | | | | | 4 | 10 | 15 | 20 | 28 | 77 | 15.4 | |
| 10 | L | 1 | 3 | 4 | 5 | 2 | 2 | 5 | 9 | 15 | 16 | 47 | 9.4 | 6.6 |
| | U | | | | | | 4 | 10 | 15 | 23 | 28 | 80 | 16 | |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | L | 1 | 3 | 5 | 2 | 4 | 2 | 5 | 11 | 12 | 16 | 46 | 9.2 | 7.4 |
| | U | | | | | | 4 | 10 | 18 | 23 | 28 | 83 | 16.6 | |
| 12 | L | 1 | 3 | 5 | 4 | 2 | 2 | 5 | 11 | 15 | 16 | 49 | 9.8 | 6.8 |
| | U | | | | | | 4 | 10 | 18 | 23 | 28 | 83 | 16.6 | |
| 13 | L | 1 | 4 | 2 | 3 | 5 | 2 | 6 | 7 | 10 | 16 | 41 | 8.2 | 6.8 |
| | U | | | | | | 4 | 9 | 14 | 20 | 28 | 75 | 15 | |
| 14 | L | 1 | 4 | 2 | 5 | 3 | 2 | 6 | 7 | 13 | 16 | 44 | 8.8 | 6.6 |
| | U | | | | | | 4 | 9 | 14 | 22 | 28 | 77 | 15.4 | |
| 15 | L | 1 | 4 | 3 | 2 | 5 | 2 | 6 | 9 | 10 | 16 | 43 | 8.6 | 6.6 |
| | U | | | | | | 4 | 9 | 15 | 20 | 28 | 76 | 15.2 | |
| 16 | L | 1 | 4 | 3 | 5 | 2 | 2 | 6 | 9 | 15 | 16 | 48 | 9.6 | 6.2 |
| | U | | | | | | 4 | 9 | 15 | 23 | 28 | 79 | 15.8 | |
| 17 | L | 1 | 4 | 5 | 2 | 3 | 2 | 6 | 12 | 13 | 16 | 49 | 9.8 | 6.2 |
| | U | | | | | | 4 | 9 | 17 | 22 | 28 | 80 | 16 | |
| 18 | L | 1 | 4 | 5 | 3 | 2 | 2 | 6 | 12 | 15 | 16 | 51 | 10.2 | 6 |
| | U | | | | | | 4 | 9 | 17 | 23 | 28 | 81 | 16.2 | |
| 19 | L | 1 | 5 | 2 | 3 | 4 | 2 | 8 | 9 | 12 | 16 | 47 | 9.4 | 7.4 |
| | U | | | | | | 4 | 12 | 17 | 23 | 28 | 84 | 16.8 | |
| 20 | L | 1 | 5 | 2 | 4 | 3 | 2 | 8 | 9 | 13 | 16 | 48 | 9.6 | 7 |
| | U | | | | | | 4 | 12 | 17 | 22 | 28 | 83 | 16.6 | |
| 21 | L | 1 | 5 | 3 | 2 | 4 | 2 | 8 | 11 | 12 | 16 | 49 | 9.8 | 7.2 |
| | U | | | | | | 4 | 12 | 18 | 23 | 28 | 85 | 17 | |
| 22 | L | 1 | 5 | 3 | 4 | 2 | 2 | 8 | 11 | 15 | 16 | 52 | 10.4 | 6.6 |
| | U | | | | | | 4 | 12 | 18 | 23 | 28 | 85 | 17 | |
| 23 | L | 1 | 5 | 4 | 2 | 3 | 2 | 8 | 12 | 13 | 16 | 51 | 10.2 | 6.4 |
| | U | | | | | | 4 | 12 | 17 | 22 | 28 | 83 | 16.6 | |
| 24 | L | 1 | 5 | 4 | 3 | 2 | 2 | 8 | 12 | 15 | 16 | 53 | 10.6 | 6.2 |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | U | | | | | | 4 | 12 | 17 | 23 | 28 | 84 | 16.8 | |
| 25 | L | 2 | 1 | 3 | 4 | 5 | 1 | 3 | 6 | 10 | 16 | 36 | 7.2 | 8.2 |
| | U | | | | | | 5 | 9 | 15 | 20 | 28 | 77 | 15.4 | |
| 26 | L | 2 | 1 | 3 | 5 | 4 | 1 | 3 | 6 | 12 | 16 | 38 | 7.6 | 8.4 |
| | U | | | | | | 5 | 9 | 15 | 23 | 28 | 80 | 16 | |
| 27 | L | 2 | 1 | 4 | 3 | 5 | 1 | 3 | 7 | 10 | 16 | 37 | 7.4 | 7.8 |
| | U | | | | | | 5 | 9 | 14 | 20 | 28 | 76 | 15.2 | |
| 28 | L | 2 | 1 | 4 | 5 | 3 | 1 | 3 | 7 | 13 | 16 | 40 | 8 | 7.6 |
| | U | | | | | | 5 | 9 | 14 | 22 | 28 | 78 | 15.6 | |
| 29 | L | 2 | 1 | 5 | 3 | 4 | 1 | 3 | 9 | 12 | 16 | 41 | 8.2 | 8.2 |
| | U | | | | | | 5 | 9 | 17 | 23 | 28 | 82 | 16.4 | |
| 30 | L | 2 | 1 | 5 | 4 | 3 | 1 | 3 | 9 | 13 | 16 | 42 | 8.4 | 7.8 |
| | U | | | | | | 5 | 9 | 17 | 22 | 28 | 81 | 16.2 | |
| 31 | L | 2 | 3 | 1 | 4 | 5 | 1 | 4 | 6 | 10 | 16 | 37 | 7.4 | 8.4 |
| | U | | | | | | 5 | 11 | 15 | 20 | 28 | 79 | 15.8 | |
| 32 | L | 2 | 3 | 1 | 5 | 4 | 1 | 4 | 6 | 12 | 16 | 39 | 7.8 | 8.6 |
| | U | | | | | | 5 | 11 | 15 | 23 | 28 | 82 | 16.4 | |
| 33 | L | 2 | 3 | 4 | 1 | 5 | 1 | 4 | 8 | 10 | 16 | 39 | 7.8 | 8.2 |
| | U | | | | | | 5 | 11 | 16 | 20 | 28 | 80 | 16 | |
| 34 | L | 2 | 3 | 4 | 5 | 1 | 1 | 4 | 8 | 14 | 16 | 43 | 8.6 | 8.2 |
| | U | | | | | | 5 | 11 | 16 | 24 | 28 | 84 | 16.8 | |
| 35 | L | 2 | 3 | 5 | 1 | 4 | 1 | 4 | 10 | 12 | 16 | 43 | 8.6 | 8.6 |
| | U | | | | | | 5 | 11 | 19 | 23 | 28 | 86 | 17.2 | |
| 36 | L | 2 | 3 | 5 | 4 | 1 | 1 | 4 | 10 | 14 | 16 | 45 | 9 | 8.4 |
| | U | | | | | | 5 | 11 | 19 | 24 | 28 | 87 | 17.4 | |
| 37 | L | 2 | 4 | 1 | 3 | 5 | 1 | 5 | 7 | 10 | 16 | 39 | 7.8 | 7.6 |
| | U | | | | | | 5 | 10 | 14 | 20 | 28 | 77 | 15.4 | |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 38 | L | 2 | 4 | 1 | 5 | 3 | 1 | 5 | 7 | 13 | 16 | 42 | 8.4 | 7.4 |
| | U | | | | | | 5 | 10 | 14 | 22 | 28 | 79 | 15.8 | |
| 39 | L | 2 | 4 | 3 | 1 | 5 | 1 | 5 | 8 | 10 | 16 | 40 | 8 | 7.8 |
| | U | | | | | | 5 | 10 | 16 | 20 | 28 | 79 | 15.8 | |
| 40 | L | 2 | 4 | 3 | 5 | 1 | 1 | 5 | 8 | 14 | 16 | 44 | 8.8 | 7.8 |
| | U | | | | | | 5 | 10 | 16 | 24 | 28 | 83 | 16.6 | |
| 41 | L | 2 | 4 | 5 | 1 | 3 | 1 | 5 | 11 | 13 | 16 | 46 | 9.2 | 7.4 |
| | U | | | | | | 5 | 10 | 18 | 22 | 28 | 83 | 16.6 | |
| 42 | L | 2 | 4 | 5 | 3 | 1 | 1 | 5 | 11 | 14 | 16 | 47 | 9.4 | 7.6 |
| | U | | | | | | 5 | 10 | 18 | 24 | 28 | 85 | 17 | |
| 43 | L | 2 | 5 | 1 | 3 | 4 | 1 | 7 | 9 | 12 | 16 | 45 | 9 | 8.2 |
| | U | | | | | | 5 | 13 | 17 | 23 | 28 | 86 | 17.2 | |
| 44 | L | 2 | 5 | 1 | 4 | 3 | 1 | 7 | 9 | 13 | 16 | 46 | 9.2 | 7.8 |
| | U | | | | | | 5 | 13 | 17 | 22 | 28 | 85 | 17 | |
| 45 | L | 2 | 5 | 3 | 1 | 4 | 1 | 7 | 10 | 12 | 16 | 46 | 9.2 | 8.4 |
| | U | | | | | | 5 | 13 | 19 | 23 | 28 | 88 | 17.6 | |
| 46 | L | 2 | 5 | 3 | 4 | 1 | 1 | 7 | 10 | 14 | 16 | 48 | 9.6 | 8.2 |
| | U | | | | | | 5 | 13 | 19 | 24 | 28 | 89 | 17.8 | |
| 47 | L | 2 | 5 | 4 | 1 | 3 | 1 | 7 | 11 | 13 | 16 | 48 | 9.6 | 7.6 |
| | U | | | | | | 5 | 13 | 18 | 22 | 28 | 86 | 17.2 | |
| 48 | L | 2 | 5 | 4 | 3 | 1 | 1 | 7 | 11 | 14 | 16 | 49 | 9.8 | 7.8 |
| | U | | | | | | 5 | 13 | 18 | 24 | 28 | 88 | 17.6 | |
| 49 | L | 3 | 1 | 2 | 4 | 5 | 3 | 5 | 6 | 10 | 16 | 40 | 8 | 7.8 |
| | U | | | | | | 6 | 10 | 15 | 20 | 28 | 79 | 15.8 | |
| 50 | L | 3 | 1 | 2 | 5 | 4 | 3 | 5 | 6 | 12 | 16 | 42 | 8.4 | 8 |
| | U | | | | | | 6 | 10 | 15 | 23 | 28 | 82 | 16.4 | |
| 51 | L | 3 | 1 | 4 | 2 | 5 | 3 | 5 | 9 | 10 | 16 | 43 | 8.6 | 7.2 |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | U | | | | | | 6 | 10 | 15 | 20 | 28 | 79 | 15.8 | |
| 52 | L | 3 | 1 | 4 | 5 | 2 | 3 | 5 | 9 | 15 | 16 | 48 | 9.6 | 6.8 |
| | U | | | | | | 6 | 10 | 15 | 23 | 28 | 82 | 16.4 | |
| 53 | L | 3 | 1 | 5 | 2 | 4 | 3 | 5 | 11 | 12 | 16 | 47 | 9.4 | 7.6 |
| | U | | | | | | 6 | 10 | 18 | 23 | 28 | 85 | 17 | |
| 54 | L | 3 | 1 | 5 | 4 | 2 | 3 | 5 | 11 | 15 | 16 | 50 | 10 | 7 |
| | U | | | | | | 6 | 10 | 18 | 23 | 28 | 85 | 17 | |
| 55 | L | 3 | 2 | 1 | 4 | 5 | 3 | 4 | 6 | 10 | 16 | 39 | 7.8 | 8.2 |
| | U | | | | | | 6 | 11 | 15 | 20 | 28 | 80 | 16 | |
| 56 | L | 3 | 2 | 1 | 5 | 4 | 3 | 4 | 6 | 12 | 16 | 41 | 8.2 | 8.4 |
| | U | | | | | | 6 | 11 | 15 | 23 | 28 | 83 | 16.6 | |
| 57 | L | 3 | 2 | 4 | 1 | 5 | 3 | 4 | 8 | 10 | 16 | 41 | 8.2 | 8 |
| | U | | | | | | 6 | 11 | 16 | 20 | 28 | 81 | 16.2 | |
| 58 | L | 3 | 2 | 4 | 5 | 1 | 3 | 4 | 8 | 14 | 16 | 45 | 9 | 8 |
| | U | | | | | | 6 | 11 | 16 | 24 | 28 | 85 | 17 | |
| 59 | L | 3 | 2 | 5 | 1 | 4 | 3 | 4 | 10 | 12 | 16 | 45 | 9 | 8.4 |
| | U | | | | | | 6 | 11 | 19 | 23 | 28 | 87 | 17.4 | |
| 60 | L | 3 | 2 | 5 | 4 | 1 | 3 | 4 | 10 | 14 | 16 | 47 | 9.4 | 8.2 |
| | U | | | | | | 6 | 11 | 19 | 24 | 28 | 88 | 17.6 | |
| 61 | L | 3 | 4 | 1 | 2 | 5 | 3 | 7 | 9 | 10 | 16 | 45 | 9 | 7 |
| | U | | | | | | 6 | 11 | 15 | 20 | 28 | 80 | 16 | |
| 62 | L | 3 | 4 | 1 | 5 | 2 | 3 | 7 | 9 | 15 | 16 | 50 | 10 | 6.6 |
| | U | | | | | | 6 | 11 | 15 | 23 | 28 | 83 | 16.6 | |
| 63 | L | 3 | 4 | 2 | 1 | 5 | 3 | 7 | 8 | 10 | 16 | 44 | 8.8 | 7.4 |
| | U | | | | | | 6 | 11 | 16 | 20 | 28 | 81 | 16.2 | |
| 64 | L | 3 | 4 | 2 | 5 | 1 | 3 | 7 | 8 | 14 | 16 | 48 | 9.6 | 7.4 |
| | U | | | | | | 6 | 11 | 16 | 24 | 28 | 85 | 17 | |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 65 | L | 3 | 4 | 5 | 1 | 2 | 3 | 7 | 13 | 15 | 16 | 54 | 10.8 | 6.6 |
| | U | | | | | | 6 | 11 | 19 | 23 | 28 | 87 | 17.4 | |
| 66 | L | 3 | 4 | 5 | 2 | 1 | 3 | 7 | 13 | 14 | 16 | 53 | 10.6 | 7 |
| | U | | | | | | 6 | 11 | 19 | 24 | 28 | 88 | 17.6 | |
| 67 | L | 3 | 5 | 1 | 2 | 4 | 3 | 9 | 11 | 12 | 16 | 51 | 10.2 | 7.6 |
| | U | | | | | | 6 | 14 | 18 | 23 | 28 | 89 | 17.8 | |
| 68 | L | 3 | 5 | 1 | 4 | 2 | 3 | 9 | 11 | 15 | 16 | 54 | 10.8 | 7 |
| | U | | | | | | 6 | 14 | 18 | 23 | 28 | 89 | 17.8 | |
| 69 | L | 3 | 5 | 2 | 1 | 4 | 3 | 9 | 10 | 12 | 16 | 50 | 10 | 8 |
| | U | | | | | | 6 | 14 | 19 | 23 | 28 | 90 | 18 | |
| 70 | L | 3 | 5 | 2 | 4 | 1 | 3 | 9 | 10 | 14 | 16 | 52 | 10.4 | 7.8 |
| | U | | | | | | 6 | 14 | 19 | 24 | 28 | 91 | 18.2 | |
| 71 | L | 3 | 5 | 4 | 1 | 2 | 3 | 9 | 13 | 15 | 16 | 56 | 11.2 | 6.8 |
| | U | | | | | | 6 | 14 | 19 | 23 | 28 | 90 | 18 | |
| 72 | L | 3 | 5 | 4 | 2 | 1 | 3 | 9 | 13 | 14 | 16 | 55 | 11 | 7.2 |
| | U | | | | | | 6 | 14 | 19 | 24 | 28 | 91 | 18.2 | |
| 73 | L | 4 | 1 | 2 | 3 | 5 | 4 | 6 | 7 | 10 | 16 | 43 | 8.6 | 6.6 |
| | U | | | | | | 5 | 9 | 14 | 20 | 28 | 76 | 15.2 | |
| 74 | L | 4 | 1 | 2 | 5 | 3 | 4 | 6 | 7 | 13 | 16 | 46 | 9.2 | 6.4 |
| | U | | | | | | 5 | 9 | 14 | 22 | 28 | 78 | 15.6 | |
| 75 | L | 4 | 1 | 3 | 2 | 5 | 4 | 6 | 9 | 10 | 16 | 45 | 9 | 6.4 |
| | U | | | | | | 5 | 9 | 15 | 20 | 28 | 77 | 15.4 | |
| 76 | L | 4 | 1 | 3 | 5 | 2 | 4 | 6 | 9 | 15 | 16 | 50 | 10 | 6 |
| | U | | | | | | 5 | 9 | 15 | 23 | 28 | 80 | 16 | |
| 77 | L | 4 | 1 | 5 | 2 | 3 | 4 | 6 | 12 | 13 | 16 | 51 | 10.2 | 6 |
| | U | | | | | | 5 | 9 | 17 | 22 | 28 | 81 | 16.2 | |
| 78 | L | 4 | 1 | 5 | 3 | 2 | 4 | 6 | 12 | 15 | 16 | 53 | 10.6 | 5.8 |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | U | | | | | | 5 | 9 | 17 | 23 | 28 | 82 | 16.4 | |
| 79 | L | 4 | 2 | 1 | 3 | 5 | 4 | 5 | 7 | 10 | 16 | 42 | 8.4 | 7 |
| | U | | | | | | 5 | 10 | 14 | 20 | 28 | 77 | 15.4 | |
| 80 | L | 4 | 2 | 1 | 5 | 3 | 4 | 5 | 7 | 13 | 16 | 45 | 9 | 6.8 |
| | U | | | | | | 5 | 10 | 14 | 22 | 28 | 79 | 15.8 | |
| 81 | L | 4 | 2 | 3 | 1 | 5 | 4 | 5 | 8 | 10 | 16 | 43 | 8.6 | 7.2 |
| | U | | | | | | 5 | 10 | 16 | 20 | 28 | 79 | 15.8 | |
| 82 | L | 4 | 2 | 3 | 5 | 1 | 4 | 5 | 8 | 14 | 16 | 47 | 9.4 | 7.2 |
| | U | | | | | | 5 | 10 | 16 | 24 | 28 | 83 | 16.6 | |
| 83 | L | 4 | 2 | 5 | 1 | 3 | 4 | 5 | 11 | 13 | 16 | 49 | 9.8 | 6.8 |
| | U | | | | | | 5 | 10 | 18 | 22 | 28 | 83 | 16.6 | |
| 84 | L | 4 | 2 | 5 | 3 | 1 | 4 | 5 | 11 | 14 | 16 | 50 | 10 | 7 |
| | U | | | | | | 5 | 10 | 18 | 24 | 28 | 85 | 17 | |
| 85 | L | 4 | 3 | 1 | 2 | 5 | 4 | 7 | 9 | 10 | 16 | 46 | 9.2 | 6.6 |
| | U | | | | | | 5 | 11 | 15 | 20 | 28 | 79 | 15.8 | |
| 86 | L | 4 | 3 | 1 | 5 | 2 | 4 | 7 | 9 | 15 | 16 | 51 | 10.2 | 6.2 |
| | U | | | | | | 5 | 11 | 15 | 23 | 28 | 82 | 16.4 | |
| 87 | L | 4 | 3 | 2 | 1 | 5 | 4 | 7 | 8 | 10 | 16 | 45 | 9 | 7 |
| | U | | | | | | 5 | 11 | 16 | 20 | 28 | 80 | 16 | |
| 88 | L | 4 | 3 | 2 | 5 | 1 | 4 | 7 | 8 | 14 | 16 | 49 | 9.8 | 7 |
| | U | | | | | | 5 | 11 | 16 | 24 | 28 | 84 | 16.8 | |
| 89 | L | 4 | 3 | 5 | 1 | 2 | 4 | 7 | 13 | 15 | 16 | 55 | 11 | 6.2 |
| | U | | | | | | 5 | 11 | 19 | 23 | 28 | 86 | 17.2 | |
| 90 | L | 4 | 3 | 5 | 2 | 1 | 4 | 7 | 13 | 14 | 16 | 54 | 10.8 | 6.6 |
| | U | | | | | | 5 | 11 | 19 | 24 | 28 | 87 | 17.4 | |
| 91 | L | 4 | 5 | 1 | 2 | 3 | 4 | 10 | 12 | 13 | 16 | 55 | 11 | 6 |
| | U | | | | | | 5 | 13 | 17 | 22 | 28 | 85 | 17 | |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 92 | L | 4 | 5 | 1 | 3 | 2 | 4 | 10 | 12 | 15 | 16 | 57 | 11.4 | 5.8 |
| | U | | | | | | 5 | 13 | 17 | 23 | 28 | 86 | 17.2 | |
| 93 | L | 4 | 5 | 2 | 1 | 3 | 4 | 10 | 11 | 13 | 16 | 54 | 10.8 | 6.4 |
| | U | | | | | | 5 | 13 | 18 | 22 | 28 | 86 | 17.2 | |
| 94 | L | 4 | 5 | 2 | 3 | 1 | 4 | 10 | 11 | 14 | 16 | 55 | 11 | 6.6 |
| | U | | | | | | 5 | 13 | 18 | 24 | 28 | 88 | 17.6 | |
| 95 | L | 4 | 5 | 3 | 1 | 2 | 4 | 10 | 13 | 15 | 16 | 58 | 11.6 | 6 |
| | U | | | | | | 5 | 13 | 19 | 23 | 28 | 88 | 17.6 | |
| 96 | L | 4 | 5 | 3 | 2 | 1 | 4 | 10 | 13 | 14 | 16 | 57 | 11.4 | 6.4 |
| | U | | | | | | 5 | 13 | 19 | 24 | 28 | 89 | 17.8 | |
| 97 | L | 5 | 1 | 2 | 3 | 4 | 6 | 8 | 9 | 12 | 16 | 51 | 10.2 | 7.4 |
| | U | | | | | | 8 | 12 | 17 | 23 | 28 | 88 | 17.6 | |
| 98 | L | 5 | 1 | 2 | 4 | 3 | 6 | 8 | 9 | 13 | 16 | 52 | 10.4 | 7 |
| | U | | | | | | 8 | 12 | 17 | 22 | 28 | 87 | 17.4 | |
| 99 | L | 5 | 1 | 3 | 2 | 4 | 6 | 8 | 11 | 12 | 16 | 53 | 10.6 | 7.2 |
| | U | | | | | | 8 | 12 | 18 | 23 | 28 | 89 | 17.8 | |
| 100 | L | 5 | 1 | 3 | 4 | 2 | 6 | 8 | 11 | 15 | 16 | 56 | 11.2 | 6.6 |
| | U | | | | | | 8 | 12 | 18 | 23 | 28 | 89 | 17.8 | |
| 101 | L | 5 | 1 | 4 | 2 | 3 | 6 | 8 | 12 | 13 | 16 | 55 | 11 | 6.4 |
| | U | | | | | | 8 | 12 | 17 | 22 | 28 | 87 | 17.4 | |
| 102 | L | 5 | 1 | 4 | 3 | 2 | 6 | 8 | 12 | 15 | 16 | 57 | 11.4 | 6.2 |
| | U | | | | | | 8 | 12 | 17 | 23 | 28 | 88 | 17.6 | |
| 103 | L | 5 | 2 | 1 | 3 | 4 | 6 | 7 | 9 | 12 | 16 | 50 | 10 | 7.8 |
| | U | | | | | | 8 | 13 | 17 | 23 | 28 | 89 | 17.8 | |
| 104 | L | 5 | 2 | 1 | 4 | 3 | 6 | 7 | 9 | 13 | 16 | 51 | 10.2 | 7.4 |
| | U | | | | | | 8 | 13 | 17 | 22 | 28 | 88 | 17.6 | |
| 105 | L | 5 | 2 | 3 | 1 | 4 | 6 | 7 | 10 | 12 | 16 | 51 | 10.2 | 8 |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | U | | | | | | 8 | 13 | 19 | 23 | 28 | 91 | 18.2 | |
| 106 | L | 5 | 2 | 3 | 4 | 1 | 6 | 7 | 10 | 14 | 16 | 53 | 10.6 | 7.8 |
| | U | | | | | | 8 | 13 | 19 | 24 | 28 | 92 | 18.4 | |
| 107 | L | 5 | 2 | 4 | 1 | 3 | 6 | 7 | 11 | 13 | 16 | 53 | 10.6 | 7.2 |
| | U | | | | | | 8 | 13 | 18 | 22 | 28 | 89 | 17.8 | |
| 108 | L | 5 | 2 | 4 | 3 | 1 | 6 | 7 | 11 | 14 | 16 | 54 | 10.8 | 7.4 |
| | U | | | | | | 8 | 13 | 18 | 24 | 28 | 91 | 18.2 | |
| 109 | L | 5 | 3 | 1 | 2 | 4 | 6 | 9 | 11 | 12 | 16 | 54 | 10.8 | 7.4 |
| | U | | | | | | 8 | 14 | 18 | 23 | 28 | 91 | 18.2 | |
| 110 | L | 5 | 3 | 1 | 4 | 2 | 6 | 9 | 11 | 15 | 16 | 57 | 11.4 | 6.8 |
| | U | | | | | | 8 | 14 | 18 | 23 | 28 | 91 | 18.2 | |
| 111 | L | 5 | 3 | 2 | 1 | 4 | 6 | 9 | 10 | 12 | 16 | 53 | 10.6 | 7.8 |
| | U | | | | | | 8 | 14 | 19 | 23 | 28 | 92 | 18.4 | |
| 112 | L | 5 | 3 | 2 | 4 | 1 | 6 | 9 | 10 | 14 | 16 | 55 | 11 | 7.6 |
| | U | | | | | | 8 | 14 | 19 | 24 | 28 | 93 | 18.6 | |
| 113 | L | 5 | 3 | 4 | 1 | 2 | 6 | 9 | 13 | 15 | 16 | 59 | 11.8 | 6.6 |
| | U | | | | | | 8 | 14 | 19 | 23 | 28 | 92 | 18.4 | |
| 114 | L | 5 | 3 | 4 | 2 | 1 | 6 | 9 | 13 | 14 | 16 | 58 | 11.6 | 7 |
| | U | | | | | | 8 | 14 | 19 | 24 | 28 | 93 | 18.6 | |
| 115 | L | 5 | 4 | 1 | 2 | 3 | 6 | 10 | 12 | 13 | 16 | 57 | 11.4 | 6.2 |
| | U | | | | | | 8 | 13 | 17 | 22 | 28 | 88 | 17.6 | |
| 116 | L | 5 | 4 | 1 | 3 | 2 | 6 | 10 | 12 | 15 | 16 | 59 | 11.8 | 6 |
| | U | | | | | | 8 | 13 | 17 | 23 | 28 | 89 | 17.8 | |
| 117 | L | 5 | 4 | 2 | 1 | 3 | 6 | 10 | 11 | 13 | 16 | 56 | 11.2 | 6.6 |
| | U | | | | | | 8 | 13 | 18 | 22 | 28 | 89 | 17.8 | |
| 118 | L | 5 | 4 | 2 | 3 | 1 | 6 | 10 | 11 | 14 | 16 | 57 | 11.4 | 6.8 |
| | U | | | | | | 8 | 13 | 18 | 24 | 28 | 91 | 18.2 | |

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 119 | L | 5 | 4 | 3 | 1 | 2 | 6 | 10 | 13 | 15 | 16 | 60 | 12 | 6.2 |
| | U | | | | | | 8 | 13 | 19 | 23 | 28 | 91 | 18.2 | |
| 120 | L | 5 | 4 | 3 | 2 | 1 | 6 | 10 | 13 | 14 | 16 | 59 | 11.8 | 6.6 |
| | U | | | | | | 8 | 13 | 19 | 24 | 28 | 92 | 18.4 | |

# Appendix B

**Solution Method for the Early-Tardy Problem using MATLAB**

One of the solutions methods is by finding all feasible schedules, and then choosing the one that gives the minimum objective value. MATLAB was used to find all possible sequences by considering the processing times' estimates (lower and upper bounds). The program that was used is:

```
function [S1, S2, C1, C2, ET1, ET2, TET1, TET2] = sch2(A, d)
numofent = size(A, 2);
S1 = perms(A(1, :));
for i = 1:factorial(numofent)
   C1(i, 1) = S1(i, 1);
   ET1(i, 1) = abs(C1(i, 1) - d);
   for j = 2:numofent
      C1(i, j) = S1(i, j) + C1(i, j-1);
      ET1(i, j) = abs(C1(i, j) - d);
   end
   TET1(i, 1) = sum(ET1(:, :));
end
S2 = perms(A(2, :));
for i = 1:factorial(numofent)
   C2(i, 1) = S2(i, 1);
   ET2(i, 1) = abs(C2(i, 1) - d);
   for j = 2:numofent
      C2(i, j) = S2(i, j) + C2(i, j-1);
      ET2(i, j) = abs(C2(i, j) - d);
   end
   TET2(i, 1) = sum(ET2(:, :));
end
```

To illustrate the use of this program, all possible schedules are generated and evaluated for the following problem (which was discussed in chapter 4):

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **J** | **1** | **2** | **3** | **4** | **5** | | | | | | | | | | | |
| **LB** | 2 | 1 | 3 | 4 | 6 | | | | | | | | | | | |
| **UB** | 4 | 5 | 6 | 5 | 8 | | | | | | | | | | | |
| **R** | 2 | 4 | 3 | 1 | 2 | | | | | | | | | | | |
| **d** | 14 | 14 | 14 | 14 | 14 | | | | | | | | | | | |
| | | | | | | | | | | | | | | | | |
| **i** | **Bound** | **Sequence of Jobs** | | | | | | | | | | | | | | |
| | | **[1]** | **[2]** | **[3]** | **[4]** | **[5]** | **$C_{[1]}$** | **$C_{[2]}$** | **$C_{[3]}$** | **$C_{[4]}$** | **$C_{[5]}$** | **$E_{[i]} + T_{[i]}$** | | | | **Total E-T** |
| 1 | L | 1 | 2 | 3 | 4 | 5 | 2 | 3 | 6 | 10 | 16 | 12 | 11 | 8 | 4 | 2 | 37 |
| | U | | | | | | 4 | 9 | 15 | 20 | 28 | 10 | 5 | 1 | 6 | 14 | 36 |
| 2 | L | 1 | 2 | 3 | 5 | 4 | 2 | 3 | 6 | 12 | 16 | 12 | 11 | 8 | 2 | 2 | 35 |
| | U | | | | | | 4 | 9 | 15 | 23 | 28 | 10 | 5 | 1 | 9 | 14 | 39 |
| 3 | L | 1 | 2 | 4 | 3 | 5 | 2 | 3 | 7 | 10 | 16 | 12 | 11 | 7 | 4 | 2 | 36 |
| | U | | | | | | 4 | 9 | 14 | 20 | 28 | 10 | 5 | 0 | 6 | 14 | 35 |
| 4 | L | 1 | 2 | 4 | 5 | 3 | 2 | 3 | 7 | 13 | 16 | 12 | 11 | 7 | 1 | 2 | 33 |
| | U | | | | | | 4 | 9 | 14 | 22 | 28 | 10 | 5 | 0 | 8 | 14 | 37 |
| 5 | L | 1 | 2 | 5 | 3 | 4 | 2 | 3 | 9 | 12 | 16 | 12 | 11 | 5 | 2 | 2 | 32 |
| | U | | | | | | 4 | 9 | 17 | 23 | 28 | 10 | 5 | 3 | 9 | 14 | 41 |
| 6 | L | 1 | 2 | 5 | 4 | 3 | 2 | 3 | 9 | 13 | 16 | 12 | 11 | 5 | 1 | 2 | 31 |
| | U | | | | | | 4 | 9 | 17 | 22 | 28 | 10 | 5 | 3 | 8 | 14 | 40 |
| 7 | L | 1 | 3 | 2 | 4 | 5 | 2 | 5 | 6 | 10 | 16 | 12 | 9 | 8 | 4 | 2 | 35 |
| | U | | | | | | 4 | 10 | 15 | 20 | 28 | 10 | 4 | 1 | 6 | 14 | 35 |
| 8 | L | 1 | 3 | 2 | 5 | 4 | 2 | 5 | 6 | 12 | 16 | 12 | 9 | 8 | 2 | 2 | 33 |
| | U | | | | | | 4 | 10 | 15 | 23 | 28 | 10 | 4 | 1 | 9 | 14 | 38 |
| 9 | L | 1 | 3 | 4 | 2 | 5 | 2 | 5 | 9 | 10 | 16 | 12 | 9 | 5 | 4 | 2 | 32 |
| | U | | | | | | 4 | 10 | 15 | 20 | 28 | 10 | 4 | 1 | 6 | 14 | 35 |

| | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 10 | L | 1 | 3 | 4 | 5 | 2 | 2 | 5 | 9 | 15 | 16 | 12 | 9 | 5 | 1 | 2 | 29 |
| | U | | | | | | 4 | 10 | 15 | 23 | 28 | 10 | 4 | 1 | 9 | 14 | 38 |
| 11 | L | 1 | 3 | 5 | 2 | 4 | 2 | 5 | 11 | 12 | 16 | 12 | 9 | 3 | 2 | 2 | 28 |
| | U | | | | | | 4 | 10 | 18 | 23 | 28 | 10 | 4 | 4 | 9 | 14 | 41 |
| 12 | L | 1 | 3 | 5 | 4 | 2 | 2 | 5 | 11 | 15 | 16 | 12 | 9 | 3 | 1 | 2 | 27 |
| | U | | | | | | 4 | 10 | 18 | 23 | 28 | 10 | 4 | 4 | 9 | 14 | 41 |
| 13 | L | 1 | 4 | 2 | 3 | 5 | 2 | 6 | 7 | 10 | 16 | 12 | 8 | 7 | 4 | 2 | 33 |
| | U | | | | | | 4 | 9 | 14 | 20 | 28 | 10 | 5 | 0 | 6 | 14 | 35 |
| 14 | L | 1 | 4 | 2 | 5 | 3 | 2 | 6 | 7 | 13 | 16 | 12 | 8 | 7 | 1 | 2 | 30 |
| | U | | | | | | 4 | 9 | 14 | 22 | 28 | 10 | 5 | 0 | 8 | 14 | 37 |
| 15 | L | 1 | 4 | 3 | 2 | 5 | 2 | 6 | 9 | 10 | 16 | 12 | 8 | 5 | 4 | 2 | 31 |
| | U | | | | | | 4 | 9 | 15 | 20 | 28 | 10 | 5 | 1 | 6 | 14 | 36 |
| 16 | L | 1 | 4 | 3 | 5 | 2 | 2 | 6 | 9 | 15 | 16 | 12 | 8 | 5 | 1 | 2 | 28 |
| | U | | | | | | 4 | 9 | 15 | 23 | 28 | 10 | 5 | 1 | 9 | 14 | 39 |
| 17 | L | 1 | 4 | 5 | 2 | 3 | 2 | 6 | 12 | 13 | 16 | 12 | 8 | 2 | 1 | 2 | 25 |
| | U | | | | | | 4 | 9 | 17 | 22 | 28 | 10 | 5 | 3 | 8 | 14 | 40 |
| 18 | L | 1 | 4 | 5 | 3 | 2 | 2 | 6 | 12 | 15 | 16 | 12 | 8 | 2 | 1 | 2 | 25 |
| | U | | | | | | 4 | 9 | 17 | 23 | 28 | 10 | 5 | 3 | 9 | 14 | 41 |
| 19 | L | 1 | 5 | 2 | 3 | 4 | 2 | 8 | 9 | 12 | 16 | 12 | 6 | 5 | 2 | 2 | 27 |
| | U | | | | | | 4 | 12 | 17 | 23 | 28 | 10 | 2 | 3 | 9 | 14 | 38 |
| 20 | L | 1 | 5 | 2 | 4 | 3 | 2 | 8 | 9 | 13 | 16 | 12 | 6 | 5 | 1 | 2 | 26 |
| | U | | | | | | 4 | 12 | 17 | 22 | 28 | 10 | 2 | 3 | 8 | 14 | 37 |
| 21 | L | 1 | 5 | 3 | 2 | 4 | 2 | 8 | 11 | 12 | 16 | 12 | 6 | 3 | 2 | 2 | 25 |
| | U | | | | | | 4 | 12 | 18 | 23 | 28 | 10 | 2 | 4 | 9 | 14 | 39 |
| 22 | L | 1 | 5 | 3 | 4 | 2 | 2 | 8 | 11 | 15 | 16 | 12 | 6 | 3 | 1 | 2 | 24 |
| | U | | | | | | 4 | 12 | 18 | 23 | 28 | 10 | 2 | 4 | 9 | 14 | 39 |
| 23 | L | 1 | 5 | 4 | 2 | 3 | 2 | 8 | 12 | 13 | 16 | 12 | 6 | 2 | 1 | 2 | 23 |

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| 24 | L | 1 | 5 | 4 | 3 | 2 | 2 | 8 | 12 | 15 | 16 | 12 | 6 | 2 | 1 | 2 | 23 |
| | U | | | | | | 4 | 12 | 17 | 23 | 28 | 10 | 2 | 3 | 9 | 14 | 38 |
| 25 | L | 2 | 1 | 3 | 4 | 5 | 1 | 3 | 6 | 10 | 16 | 13 | 11 | 8 | 4 | 2 | 38 |
| | U | | | | | | 5 | 9 | 15 | 20 | 28 | 9 | 5 | 1 | 6 | 14 | 35 |
| 26 | L | 2 | 1 | 3 | 5 | 4 | 1 | 3 | 6 | 12 | 16 | 13 | 11 | 8 | 2 | 2 | 36 |
| | U | | | | | | 5 | 9 | 15 | 23 | 28 | 9 | 5 | 1 | 9 | 14 | 38 |
| 27 | L | 2 | 1 | 4 | 3 | 5 | 1 | 3 | 7 | 10 | 16 | 13 | 11 | 7 | 4 | 2 | 37 |
| | U | | | | | | 5 | 9 | 14 | 20 | 28 | 9 | 5 | 0 | 6 | 14 | 34 |
| 28 | L | 2 | 1 | 4 | 5 | 3 | 1 | 3 | 7 | 13 | 16 | 13 | 11 | 7 | 1 | 2 | 34 |
| | U | | | | | | 5 | 9 | 14 | 22 | 28 | 9 | 5 | 0 | 8 | 14 | 36 |
| 29 | L | 2 | 1 | 5 | 3 | 4 | 1 | 3 | 9 | 12 | 16 | 13 | 11 | 5 | 2 | 2 | 33 |
| | U | | | | | | 5 | 9 | 17 | 23 | 28 | 9 | 5 | 3 | 9 | 14 | 40 |
| 30 | L | 2 | 1 | 5 | 4 | 3 | 1 | 3 | 9 | 13 | 16 | 13 | 11 | 5 | 1 | 2 | 32 |
| | U | | | | | | 5 | 9 | 17 | 22 | 28 | 9 | 5 | 3 | 8 | 14 | 39 |
| 31 | L | 2 | 3 | 1 | 4 | 5 | 1 | 4 | 6 | 10 | 16 | 13 | 10 | 8 | 4 | 2 | 37 |
| | U | | | | | | 5 | 11 | 15 | 20 | 28 | 9 | 3 | 1 | 6 | 14 | 33 |
| 32 | L | 2 | 3 | 1 | 5 | 4 | 1 | 4 | 6 | 12 | 16 | 13 | 10 | 8 | 2 | 2 | 35 |
| | U | | | | | | 5 | 11 | 15 | 23 | 28 | 9 | 3 | 1 | 9 | 14 | 36 |
| 33 | L | 2 | 3 | 4 | 1 | 5 | 1 | 4 | 8 | 10 | 16 | 13 | 10 | 6 | 4 | 2 | 35 |
| | U | | | | | | 5 | 11 | 16 | 20 | 28 | 9 | 3 | 2 | 6 | 14 | 34 |
| 34 | L | 2 | 3 | 4 | 5 | 1 | 1 | 4 | 8 | 14 | 16 | 13 | 10 | 6 | 0 | 2 | 31 |
| | U | | | | | | 5 | 11 | 16 | 24 | 28 | 9 | 3 | 2 | 10 | 14 | 38 |
| 35 | L | 2 | 3 | 5 | 1 | 4 | 1 | 4 | 10 | 12 | 16 | 13 | 10 | 4 | 2 | 2 | 31 |
| | U | | | | | | 5 | 11 | 19 | 23 | 28 | 9 | 3 | 5 | 9 | 14 | 40 |
| 36 | L | 2 | 3 | 5 | 4 | 1 | 1 | 4 | 10 | 14 | 16 | 13 | 10 | 4 | 0 | 2 | 29 |
| | U | | | | | | 5 | 11 | 19 | 24 | 28 | 9 | 3 | 5 | 10 | 14 | 41 |

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| 37 | L | 2 | 4 | 1 | 3 | 5 | 1 | 5 | 7 | 10 | 16 | 13 | 9 | 7 | 4 | 2 | 35 |
| | U | | | | | | 5 | 10 | 14 | 20 | 28 | 9 | 4 | 0 | 6 | 14 | 33 |
| 38 | L | 2 | 4 | 1 | 5 | 3 | 1 | 5 | 7 | 13 | 16 | 13 | 9 | 7 | 1 | 2 | 32 |
| | U | | | | | | 5 | 10 | 14 | 22 | 28 | 9 | 4 | 0 | 8 | 14 | 35 |
| 39 | L | 2 | 4 | 3 | 1 | 5 | 1 | 5 | 8 | 10 | 16 | 13 | 9 | 6 | 4 | 2 | 34 |
| | U | | | | | | 5 | 10 | 16 | 20 | 28 | 9 | 4 | 2 | 6 | 14 | 35 |
| 40 | L | 2 | 4 | 3 | 5 | 1 | 1 | 5 | 8 | 14 | 16 | 13 | 9 | 6 | 0 | 2 | 30 |
| | U | | | | | | 5 | 10 | 16 | 24 | 28 | 9 | 4 | 2 | 10 | 14 | 39 |
| 41 | L | 2 | 4 | 5 | 1 | 3 | 1 | 5 | 11 | 13 | 16 | 13 | 9 | 3 | 1 | 2 | 28 |
| | U | | | | | | 5 | 10 | 18 | 22 | 28 | 9 | 4 | 4 | 8 | 14 | 39 |
| 42 | L | 2 | 4 | 5 | 3 | 1 | 1 | 5 | 11 | 14 | 16 | 13 | 9 | 3 | 0 | 2 | 27 |
| | U | | | | | | 5 | 10 | 18 | 24 | 28 | 9 | 4 | 4 | 10 | 14 | 41 |
| 43 | L | 2 | 5 | 1 | 3 | 4 | 1 | 7 | 9 | 12 | 16 | 13 | 7 | 5 | 2 | 2 | 29 |
| | U | | | | | | 5 | 13 | 17 | 23 | 28 | 9 | 1 | 3 | 9 | 14 | 36 |
| 44 | L | 2 | 5 | 1 | 4 | 3 | 1 | 7 | 9 | 13 | 16 | 13 | 7 | 5 | 1 | 2 | 28 |
| | U | | | | | | 5 | 13 | 17 | 22 | 28 | 9 | 1 | 3 | 8 | 14 | 35 |
| 45 | L | 2 | 5 | 3 | 1 | 4 | 1 | 7 | 10 | 12 | 16 | 13 | 7 | 4 | 2 | 2 | 28 |
| | U | | | | | | 5 | 13 | 19 | 23 | 28 | 9 | 1 | 5 | 9 | 14 | 38 |
| 46 | L | 2 | 5 | 3 | 4 | 1 | 1 | 7 | 10 | 14 | 16 | 13 | 7 | 4 | 0 | 2 | 26 |
| | U | | | | | | 5 | 13 | 19 | 24 | 28 | 9 | 1 | 5 | 10 | 14 | 39 |
| 47 | L | 2 | 5 | 4 | 1 | 3 | 1 | 7 | 11 | 13 | 16 | 13 | 7 | 3 | 1 | 2 | 26 |
| | U | | | | | | 5 | 13 | 18 | 22 | 28 | 9 | 1 | 4 | 8 | 14 | 36 |
| 48 | L | 2 | 5 | 4 | 3 | 1 | 1 | 7 | 11 | 14 | 16 | 13 | 7 | 3 | 0 | 2 | 25 |
| | U | | | | | | 5 | 13 | 18 | 24 | 28 | 9 | 1 | 4 | 10 | 14 | 38 |
| 49 | L | 3 | 1 | 2 | 4 | 5 | 3 | 5 | 6 | 10 | 16 | 11 | 9 | 8 | 4 | 2 | 34 |
| | U | | | | | | 6 | 10 | 15 | 20 | 28 | 8 | 4 | 1 | 6 | 14 | 33 |
| 50 | L | 3 | 1 | 2 | 5 | 4 | 3 | 5 | 6 | 12 | 16 | 11 | 9 | 8 | 2 | 2 | 32 |

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| 51 | L | 3 | 1 | 4 | 2 | 5 | 3 | 5 | 9 | 10 | 16 | 11 | 9 | 5 | 4 | 2 | 31 |
| | U | | | | | | 6 | 10 | 15 | 20 | 28 | 8 | 4 | 1 | 6 | 14 | 33 |
| 52 | L | 3 | 1 | 4 | 5 | 2 | 3 | 5 | 9 | 15 | 16 | 11 | 9 | 5 | 1 | 2 | 28 |
| | U | | | | | | 6 | 10 | 15 | 23 | 28 | 8 | 4 | 1 | 9 | 14 | 36 |
| 53 | L | 3 | 1 | 5 | 2 | 4 | 3 | 5 | 11 | 12 | 16 | 11 | 9 | 3 | 2 | 2 | 27 |
| | U | | | | | | 6 | 10 | 18 | 23 | 28 | 8 | 4 | 4 | 9 | 14 | 39 |
| 54 | L | 3 | 1 | 5 | 4 | 2 | 3 | 5 | 11 | 15 | 16 | 11 | 9 | 3 | 1 | 2 | 26 |
| | U | | | | | | 6 | 10 | 18 | 23 | 28 | 8 | 4 | 4 | 9 | 14 | 39 |
| 55 | L | 3 | 2 | 1 | 4 | 5 | 3 | 4 | 6 | 10 | 16 | 11 | 10 | 8 | 4 | 2 | 35 |
| | U | | | | | | 6 | 11 | 15 | 20 | 28 | 8 | 3 | 1 | 6 | 14 | 32 |
| 56 | L | 3 | 2 | 1 | 5 | 4 | 3 | 4 | 6 | 12 | 16 | 11 | 10 | 8 | 2 | 2 | 33 |
| | U | | | | | | 6 | 11 | 15 | 23 | 28 | 8 | 3 | 1 | 9 | 14 | 35 |
| 57 | L | 3 | 2 | 4 | 1 | 5 | 3 | 4 | 8 | 10 | 16 | 11 | 10 | 6 | 4 | 2 | 33 |
| | U | | | | | | 6 | 11 | 16 | 20 | 28 | 8 | 3 | 2 | 6 | 14 | 33 |
| 58 | L | 3 | 2 | 4 | 5 | 1 | 3 | 4 | 8 | 14 | 16 | 11 | 10 | 6 | 0 | 2 | 29 |
| | U | | | | | | 6 | 11 | 16 | 24 | 28 | 8 | 3 | 2 | 10 | 14 | 37 |
| 59 | L | 3 | 2 | 5 | 1 | 4 | 3 | 4 | 10 | 12 | 16 | 11 | 10 | 4 | 2 | 2 | 29 |
| | U | | | | | | 6 | 11 | 19 | 23 | 28 | 8 | 3 | 5 | 9 | 14 | 39 |
| 60 | L | 3 | 2 | 5 | 4 | 1 | 3 | 4 | 10 | 14 | 16 | 11 | 10 | 4 | 0 | 2 | 27 |
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| | U | | | | | | 6 | 11 | 15 | 20 | 28 | 8 | 3 | 1 | 6 | 14 | 32 |
| 62 | L | 3 | 4 | 1 | 5 | 2 | 3 | 7 | 9 | 15 | 16 | 11 | 7 | 5 | 1 | 2 | 26 |
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| 64 | L | 3 | 4 | 2 | 5 | 1 | 3 | 7 | 8 | 14 | 16 | 11 | 7 | 6 | 0 | 2 | 26 |
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| 65 | L | 3 | 4 | 5 | 1 | 2 | 3 | 7 | 13 | 15 | 16 | 11 | 7 | 1 | 1 | 2 | 22 |
| | U | | | | | | 6 | 11 | 19 | 23 | 28 | 8 | 3 | 5 | 9 | 14 | 39 |
| 66 | L | 3 | 4 | 5 | 2 | 1 | 3 | 7 | 13 | 14 | 16 | 11 | 7 | 1 | 0 | 2 | 21 |
| | U | | | | | | 6 | 11 | 19 | 24 | 28 | 8 | 3 | 5 | 10 | 14 | 40 |
| 67 | L | 3 | 5 | 1 | 2 | 4 | 3 | 9 | 11 | 12 | 16 | 11 | 5 | 3 | 2 | 2 | 23 |
| | U | | | | | | 6 | 14 | 18 | 23 | 28 | 8 | 0 | 4 | 9 | 14 | 35 |
| 68 | L | 3 | 5 | 1 | 4 | 2 | 3 | 9 | 11 | 15 | 16 | 11 | 5 | 3 | 1 | 2 | 22 |
| | U | | | | | | 6 | 14 | 18 | 23 | 28 | 8 | 0 | 4 | 9 | 14 | 35 |
| 69 | L | 3 | 5 | 2 | 1 | 4 | 3 | 9 | 10 | 12 | 16 | 11 | 5 | 4 | 2 | 2 | 24 |
| | U | | | | | | 6 | 14 | 19 | 23 | 28 | 8 | 0 | 5 | 9 | 14 | 36 |
| 70 | L | 3 | 5 | 2 | 4 | 1 | 3 | 9 | 10 | 14 | 16 | 11 | 5 | 4 | 0 | 2 | 22 |
| | U | | | | | | 6 | 14 | 19 | 24 | 28 | 8 | 0 | 5 | 10 | 14 | 37 |
| 71 | L | 3 | 5 | 4 | 1 | 2 | 3 | 9 | 13 | 15 | 16 | 11 | 5 | 1 | 1 | 2 | 20 |
| | U | | | | | | 6 | 14 | 19 | 23 | 28 | 8 | 0 | 5 | 9 | 14 | 36 |
| 72 | L | 3 | 5 | 4 | 2 | 1 | 3 | 9 | 13 | 14 | 16 | 11 | 5 | 1 | 0 | 2 | 19 |
| | U | | | | | | 6 | 14 | 19 | 24 | 28 | 8 | 0 | 5 | 10 | 14 | 37 |
| 73 | L | 4 | 1 | 2 | 3 | 5 | 4 | 6 | 7 | 10 | 16 | 10 | 8 | 7 | 4 | 2 | 31 |
| | U | | | | | | 5 | 9 | 14 | 20 | 28 | 9 | 5 | 0 | 6 | 14 | 34 |
| 74 | L | 4 | 1 | 2 | 5 | 3 | 4 | 6 | 7 | 13 | 16 | 10 | 8 | 7 | 1 | 2 | 28 |
| | U | | | | | | 5 | 9 | 14 | 22 | 28 | 9 | 5 | 0 | 8 | 14 | 36 |
| 75 | L | 4 | 1 | 3 | 2 | 5 | 4 | 6 | 9 | 10 | 16 | 10 | 8 | 5 | 4 | 2 | 29 |
| | U | | | | | | 5 | 9 | 15 | 20 | 28 | 9 | 5 | 1 | 6 | 14 | 35 |
| 76 | L | 4 | 1 | 3 | 5 | 2 | 4 | 6 | 9 | 15 | 16 | 10 | 8 | 5 | 1 | 2 | 26 |
| | U | | | | | | 5 | 9 | 15 | 23 | 28 | 9 | 5 | 1 | 9 | 14 | 38 |
| 77 | L | 4 | 1 | 5 | 2 | 3 | 4 | 6 | 12 | 13 | 16 | 10 | 8 | 2 | 1 | 2 | 23 |

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| 78 | L | 4 | 1 | 5 | 3 | 2 | 4 | 6 | 12 | 15 | 16 | 10 | 8 | 2 | 1 | 2 | 23 |
| | U | | | | | | 5 | 9 | 17 | 23 | 28 | 9 | 5 | 3 | 9 | 14 | 40 |
| 79 | L | 4 | 2 | 1 | 3 | 5 | 4 | 5 | 7 | 10 | 16 | 10 | 9 | 7 | 4 | 2 | 32 |
| | U | | | | | | 5 | 10 | 14 | 20 | 28 | 9 | 4 | 0 | 6 | 14 | 33 |
| 80 | L | 4 | 2 | 1 | 5 | 3 | 4 | 5 | 7 | 13 | 16 | 10 | 9 | 7 | 1 | 2 | 29 |
| | U | | | | | | 5 | 10 | 14 | 22 | 28 | 9 | 4 | 0 | 8 | 14 | 35 |
| 81 | L | 4 | 2 | 3 | 1 | 5 | 4 | 5 | 8 | 10 | 16 | 10 | 9 | 6 | 4 | 2 | 31 |
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| 82 | L | 4 | 2 | 3 | 5 | 1 | 4 | 5 | 8 | 14 | 16 | 10 | 9 | 6 | 0 | 2 | 27 |
| | U | | | | | | 5 | 10 | 16 | 24 | 28 | 9 | 4 | 2 | 10 | 14 | 39 |
| 83 | L | 4 | 2 | 5 | 1 | 3 | 4 | 5 | 11 | 13 | 16 | 10 | 9 | 3 | 1 | 2 | 25 |
| | U | | | | | | 5 | 10 | 18 | 22 | 28 | 9 | 4 | 4 | 8 | 14 | 39 |
| 84 | L | 4 | 2 | 5 | 3 | 1 | 4 | 5 | 11 | 14 | 16 | 10 | 9 | 3 | 0 | 2 | 24 |
| | U | | | | | | 5 | 10 | 18 | 24 | 28 | 9 | 4 | 4 | 10 | 14 | 41 |
| 85 | L | 4 | 3 | 1 | 2 | 5 | 4 | 7 | 9 | 10 | 16 | 10 | 7 | 5 | 4 | 2 | 28 |
| | U | | | | | | 5 | 11 | 15 | 20 | 28 | 9 | 3 | 1 | 6 | 14 | 33 |
| 86 | L | 4 | 3 | 1 | 5 | 2 | 4 | 7 | 9 | 15 | 16 | 10 | 7 | 5 | 1 | 2 | 25 |
| | U | | | | | | 5 | 11 | 15 | 23 | 28 | 9 | 3 | 1 | 9 | 14 | 36 |
| 87 | L | 4 | 3 | 2 | 1 | 5 | 4 | 7 | 8 | 10 | 16 | 10 | 7 | 6 | 4 | 2 | 29 |
| | U | | | | | | 5 | 11 | 16 | 20 | 28 | 9 | 3 | 2 | 6 | 14 | 34 |
| 88 | L | 4 | 3 | 2 | 5 | 1 | 4 | 7 | 8 | 14 | 16 | 10 | 7 | 6 | 0 | 2 | 25 |
| | U | | | | | | 5 | 11 | 16 | 24 | 28 | 9 | 3 | 2 | 10 | 14 | 38 |
| 89 | L | 4 | 3 | 5 | 1 | 2 | 4 | 7 | 13 | 15 | 16 | 10 | 7 | 1 | 1 | 2 | 21 |
| | U | | | | | | 5 | 11 | 19 | 23 | 28 | 9 | 3 | 5 | 9 | 14 | 40 |
| 90 | L | 4 | 3 | 5 | 2 | 1 | 4 | 7 | 13 | 14 | 16 | 10 | 7 | 1 | 0 | 2 | 20 |
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| 91 | L | 4 | 5 | 1 | 2 | 3 | 4 | 10 | 12 | 13 | 16 | 10 | 4 | 2 | 1 | 2 | 19 |
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| 92 | L | 4 | 5 | 1 | 3 | 2 | 4 | 10 | 12 | 15 | 16 | 10 | 4 | 2 | 1 | 2 | 19 |
| | U | | | | | | 5 | 13 | 17 | 23 | 28 | 9 | 1 | 3 | 9 | 14 | 36 |
| 93 | L | 4 | 5 | 2 | 1 | 3 | 4 | 10 | 11 | 13 | 16 | 10 | 4 | 3 | 1 | 2 | 20 |
| | U | | | | | | 5 | 13 | 18 | 22 | 28 | 9 | 1 | 4 | 8 | 14 | 36 |
| 94 | L | 4 | 5 | 2 | 3 | 1 | 4 | 10 | 11 | 14 | 16 | 10 | 4 | 3 | 0 | 2 | 19 |
| | U | | | | | | 5 | 13 | 18 | 24 | 28 | 9 | 1 | 4 | 10 | 14 | 38 |
| 95 | L | 4 | 5 | 3 | 1 | 2 | 4 | 10 | 13 | 15 | 16 | 10 | 4 | 1 | 1 | 2 | 18 |
| | U | | | | | | 5 | 13 | 19 | 23 | 28 | 9 | 1 | 5 | 9 | 14 | 38 |
| 96 | L | 4 | 5 | 3 | 2 | 1 | 4 | 10 | 13 | 14 | 16 | 10 | 4 | 1 | 0 | 2 | 17 |
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| 97 | L | 5 | 1 | 2 | 3 | 4 | 6 | 8 | 9 | 12 | 16 | 8 | 6 | 5 | 2 | 2 | 23 |
| | U | | | | | | 8 | 12 | 17 | 23 | 28 | 6 | 2 | 3 | 9 | 14 | 34 |
| 98 | L | 5 | 1 | 2 | 4 | 3 | 6 | 8 | 9 | 13 | 16 | 8 | 6 | 5 | 1 | 2 | 22 |
| | U | | | | | | 8 | 12 | 17 | 22 | 28 | 6 | 2 | 3 | 8 | 14 | 33 |
| 99 | L | 5 | 1 | 3 | 2 | 4 | 6 | 8 | 11 | 12 | 16 | 8 | 6 | 3 | 2 | 2 | 21 |
| | U | | | | | | 8 | 12 | 18 | 23 | 28 | 6 | 2 | 4 | 9 | 14 | 35 |
| 100 | L | 5 | 1 | 3 | 4 | 2 | 6 | 8 | 11 | 15 | 16 | 8 | 6 | 3 | 1 | 2 | 20 |
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| 101 | L | 5 | 1 | 4 | 2 | 3 | 6 | 8 | 12 | 13 | 16 | 8 | 6 | 2 | 1 | 2 | 19 |
| | U | | | | | | 8 | 12 | 17 | 22 | 28 | 6 | 2 | 3 | 8 | 14 | 33 |
| 102 | L | 5 | 1 | 4 | 3 | 2 | 6 | 8 | 12 | 15 | 16 | 8 | 6 | 2 | 1 | 2 | 19 |
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| 104 | L | 5 | 2 | 1 | 4 | 3 | 6 | 7 | 9 | 13 | 16 | 8 | 7 | 5 | 1 | 2 | 23 |

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| | U | | | | | | 8 | 13 | 19 | 23 | 28 | 6 | 1 | 5 | 9 | 14 | 35 |
| 106 | L | 5 | 2 | 3 | 4 | 1 | 6 | 7 | 10 | 14 | 16 | 8 | 7 | 4 | 0 | 2 | 21 |
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| | U | | | | | | 8 | 13 | 18 | 22 | 28 | 6 | 1 | 4 | 8 | 14 | 33 |
| 108 | L | 5 | 2 | 4 | 3 | 1 | 6 | 7 | 11 | 14 | 16 | 8 | 7 | 3 | 0 | 2 | 20 |
| | U | | | | | | 8 | 13 | 18 | 24 | 28 | 6 | 1 | 4 | 10 | 14 | 35 |
| 109 | L | 5 | 3 | 1 | 2 | 4 | 6 | 9 | 11 | 12 | 16 | 8 | 5 | 3 | 2 | 2 | 20 |
| | U | | | | | | 8 | 14 | 18 | 23 | 28 | 6 | 0 | 4 | 9 | 14 | 33 |
| 110 | L | 5 | 3 | 1 | 4 | 2 | 6 | 9 | 11 | 15 | 16 | 8 | 5 | 3 | 1 | 2 | 19 |
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| 111 | L | 5 | 3 | 2 | 1 | 4 | 6 | 9 | 10 | 12 | 16 | 8 | 5 | 4 | 2 | 2 | 21 |
| | U | | | | | | 8 | 14 | 19 | 23 | 28 | 6 | 0 | 5 | 9 | 14 | 34 |
| 112 | L | 5 | 3 | 2 | 4 | 1 | 6 | 9 | 10 | 14 | 16 | 8 | 5 | 4 | 0 | 2 | 19 |
| | U | | | | | | 8 | 14 | 19 | 24 | 28 | 6 | 0 | 5 | 10 | 14 | 35 |
| 113 | L | 5 | 3 | 4 | 1 | 2 | 6 | 9 | 13 | 15 | 16 | 8 | 5 | 1 | 1 | 2 | 17 |
| | U | | | | | | 8 | 14 | 19 | 23 | 28 | 6 | 0 | 5 | 9 | 14 | 34 |
| 114 | L | 5 | 3 | 4 | 2 | 1 | 6 | 9 | 13 | 14 | 16 | 8 | 5 | 1 | 0 | 2 | 16 |
| | U | | | | | | 8 | 14 | 19 | 24 | 28 | 6 | 0 | 5 | 10 | 14 | 35 |
| 115 | L | 5 | 4 | 1 | 2 | 3 | 6 | 10 | 12 | 13 | 16 | 8 | 4 | 2 | 1 | 2 | 17 |
| | U | | | | | | 8 | 13 | 17 | 22 | 28 | 6 | 1 | 3 | 8 | 14 | 32 |
| 116 | L | 5 | 4 | 1 | 3 | 2 | 6 | 10 | 12 | 15 | 16 | 8 | 4 | 2 | 1 | 2 | 17 |
| | U | | | | | | 8 | 13 | 17 | 23 | 28 | 6 | 1 | 3 | 9 | 14 | 33 |
| 117 | L | 5 | 4 | 2 | 1 | 3 | 6 | 10 | 11 | 13 | 16 | 8 | 4 | 3 | 1 | 2 | 18 |
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| 118 | L | 5 | 4 | 2 | 3 | 1 | 6 | 10 | 11 | 14 | 16 | 8 | 4 | 3 | 0 | 2 | 17 |
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| 119 | L | 5 | 4 | 3 | 1 | 2 | 6 | 10 | 13 | 15 | 16 | 8 | 4 | 1 | 1 | 2 | 16 |
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| 120 | L | 5 | 4 | 3 | 2 | 1 | 6 | 10 | 13 | 14 | 16 | 8 | 4 | 1 | 0 | 2 | 15 |
| | U | | | | | | 8 | 13 | 19 | 24 | 28 | 6 | 1 | 5 | 10 | 14 | 36 |

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